Hausdorff School

“Uniformity and Stability of Oscillatory Integrals”

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organized by
Philip Gressman, Dóminique Kemp, Lillian Pierce

Abstracts

Michael Christ (University of California)

On multilinear implicitly oscillatory integrals

Abstract: An archetypal oscillatory integral inequality states that

$$\Big| \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\, g(y)\, e^{i\lambda\psi(x,y)}\, \eta(x,y)\, dx\, dy \Big| \le C|\lambda|^{-\tau} \|f\|_{L^2} \|g\|_{L^2}$$

where $\lambda \in \mathbb{R}$ is a large parameter, the real-valued phase function ψ is smooth and is nondegenerate in a suitable sense, f, g are arbitrary L^2 functions, η is a smooth compactly supported cutoff function, and $\tau > 0$ and $C < \infty$ depend on ψ but not on f, g, λ. Its main features are the decaying factor $|\lambda|^{-\tau}$, the absence of any smoothness hypothesis on the measurable factors f, g, and the interplay between the structure of ψ and the product structure of $f(x)\, g(y)$. If ψ is nonconstant then $e^{i\lambda\psi}$ oscillates rapidly, creating cancellation that potentially results in smallness of the integral.
This course is concerned with related inequalities that are multilinear rather than bilinear, are singular in a certain respect, and have an implicitly oscillatory character. One of the main issues is the relevant notion of nondegeneracy.
Implicitly oscillatory integrals involve no overtly oscillatory factor $e^{i\lambda\psi}$; instead, the measurable factors f_j are themselves assumed to be oscillatory, but in a less structured way. A typical integral of this type takes the form

$$\int_{\mathbb{R}^2} \prod_{j=1}^{3} (f_j \circ \varphi_j)(x)\, \eta(x)\, dx$$

where $\varphi_j : \mathbb{R}^2 \to \mathbb{R}^1$ are smooth submersions and the functions f_j are merely measurable. The desired upper bound is expressed in terms of strictly negative order Sobolev norms of these functions.
These integrals arise in connection with multilinear maximal functions in harmonic analysis, with the theory of weak limits, and with Ramsey theory. There are connections with web geometry and with sublevel set inequalities, which are a key element of the proofs.
These lectures will begin with a quick review of some widely known results. Certain (explicitly) multilinear oscillatory inequalties will be stated, and the proof of one will be discussed in detail. Inequalities for implicitly oscillatory integrals will then be introduced. Proofs for trilinear iunequalities will be outlined, with some key steps presented in detail. Some connections will be explained. A few open questions will be highlighted.

Familiarity with measure and integration and multivariable calculus will be assumed, but only rather basic Fourier analysis will be required.

Polona Durcik (Chapman University)

Estimates in harmonic analysis with applications to ergodic averages

Abstract: We will discuss how some estimates for oscillatory and singular integrals are used to prove convergence results for various types of ergodic averages. One such estimate is a trilinear smoothing inequality and its application is to a.e. convergence of continuous-time quadratic averages with respect to two commuting R-actions. Other estimates that we will discuss are Lp bounds for a class of singular Brascamp-Lieb integrals, which can be used to show quantitative results on norm convergence of double and triple ergodic averages for commuting transformations, by proving variation estimates in the norm. This talk is based on joint works with M. Christ, V. Kovac, J. Roos, K. Skreb, L. Slavikova, and C. Thiele.

Philip Gressman (University of Pennsylvania)

An Introduction to Oscillatory Integrals

Abstract: This talk series will review some of the landmark results and most important things to know about the behavior of oscillatory integrals. Beginning with sublevel sets and van der Corput's lemma, a broad goal will be to see why the problem estimating oscillatory integrals is so much more difficult in several variables as opposed to one and why faster decay can be harder to understand than slower decay. No prior experience working with oscillatory integrals will be assumed

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Dóminique Kemp (UW Madison/IAS/Princeton)

Restriction over Frenét boxes and applications to maximal averages over space curves

Abstract: In this talk, we present several restriction-related estimates over Frenét boxes generated by an arbitrary nondegenerate curve in R^n. Particular attention will be paid to the dimensional dependence of the inequalities and their uniformity relative to bounds on the nondegeneracy and C^{n+1} norm. Regard for the latter paves the way for developing a maximal average theory for curves exhibiting degeneracy. This has yet to be explored in dimensions 3 and greater. Connections to the yet open problem of maximal averages over curves in 4+ dimensions will be explored

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Dominique Maldague (Massachusetts Institute of Technology)

Problems in Fourier restriction theory

Abstract: This talk will provide an overview of recent developments in Fourier restriction theory, which one could describe as the study of exponential sums over restricted frequency sets with geometric structure, typically arising in PDE or number theory. Decoupling inequalities measure the square root cancellation behavior of these exponential sums. I will highlight questions from areas outside of harmonic analysis that are approachable by decoupling/Fourier restriction theory, some with satisfying answers and some which remain open.

Ian Petrow (UCL University)

Oscillatory integrals and stationary phase estimates in analytic number theory

Abstract: I will give an overview of how oscillatory integrals arise in analytic number theory, especially the theory of L-functions and automorphic forms. Usually the integrals we are faced with are multiple-yet-low dimensional, so that they are approachable by repeated one-dimensional stationary phase estimates. I will present some lemmas that allow one to pass information on the uniformity with respect to the other variables from one dimension of the stationary phase to the next, which has been useful to us in estimating oscillatory integrals one dimension at a time.

James Wright (The University of Edinburgh)

A unified theory for oscillatory integrals defined over locally compact fields

Abstract: In harmonic analysis, oscillatory integrals are a basic object of study. They arise when one computes the Fourier transform of a measure or a distribution. Traditionally euclidean harmonic analysis has had fruitful interactions with many areas, including number theory; for example, Bourgain's use of exponential sums and the circle method in the study of nonlinear Schrodinger equations. In a dramatic reversal, tools from euclidean harmonic analysis have been used to give a complete solution of the Vinogradov Mean Value Theorem, a central open problem in analytic number theory since the 1920s with profound implications in the theory of exponential sums. In these lectures we present a new and developing theory for oscillatory integrals defined over a general locally compact field. When the field is the real field, we are looking at familiar oscillatory integrals. But if we move to the p-adic setting, then complete exponential sums can be realised as oscillatory integrals over the p-adic field. When the field is finite, oscillatory integrals in this context are exponential sums over finite fields which are a basic object in algebraic geometry and play a central role in A. Weil's Riemann Hypothesis over finite field. Here we begin to develop a theory to bring these seemingly disparate objects (real oscillator integrals, character sums and exponential sums over finite fields) under a single framework. Our theory is guided by a certain scale invariant perspective. Scaling issues for oscillateory integral are intimately connected with uniformity and stability issues for oscillatory integrals. Furthermore, interesting new phenomena arise when we think about oscillatory integrals in this generality. We will highlight a certain "self-improving" mechanism built into our theory; our bounds automatically self-improve.

Hong Wang (NYU Courant Institute)

Incidence estimate and oscillatory integrals

Abstract: We will discuss some recent multi-scale method on incidence estimate for tubes and its application to oscillatory integrals. The multi-scale method was developed by Keleti-Shmerkin, Orponen, Shmerkin, etc.

Ruixiang Zhang (University of California)

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