Workshop on

"Geometric and Applied Analysis"

July 12 - 16, 2021

organized by
Tim Laux (Bonn)

Abstracts

Giovanni Alberti (University of Pisa)

Frobenius theorem for non-smooth sets and currents

Abstract: If V be a distribution of k-planes in the Euclidean space and S is a k-dimensional surface tangent to V, then Frobenius theorem states that V must be involutive at every point of S.
In this seminar I will consider some generalization of this statement to weaker notions of surfaces, such as rectifiable sets and currents.
It turns out that if S is a subset of a k-dimensional surface then the validity of the statement depends on a combination of the regularity of the boundary of S and of the surface: if the surface is of class $C^{1,\alpha}$ with α sufficiently large the no regularity is required on the boundary of S, while at the other extreme, if the surface is only of class C^1 then it is needed that the boundary of S has finite mass (as a current); in particular Frobenius theorem holds if S is an integral current. Finally, if S is normal current but is not rectifiable, then the key is a certain geometric property of the boundary of S, and this question is strictly related to the problem of decomposing S into integral/rectifiable currents.
These results are part of an ongoing research with Annalisa Massaccesi (University of Padova), Evgeni Stepanov (Steklov Institute, Saint Petersburg) and Andrea Merlo (University Paris-Saclay).

Marcello Carioni (University of Cambridge)

Superposition principles, extremal points and dynamic variational problems

Abstract: In this talk, we are concerned with the analysis of dynamic variational problems that are regularized in time by Optimal Transport energies. In particular, the evolution of time-dependent positive measures is regularized by the dynamic formulation of the Benamou-Brenier energy and the Hellinger-Kantorovich energy. We present a numerical converging algorithm to compute minimizers of the above-mentioned dynamic problems that is based on constructing a sequence of time-dependent positive measures by iteratively adding the extremal points of the unit ball of the Benamou-Brenier energy and the Hellinger-Kantorovich energy. As this approach requires to characterize such extremal points, we discuss old and new superposition principles for the homogeneous and non-homogeneous continuity equation highlighting their importance in understanding the geometry of extremal points for dynamic Optimal Transport energies.

Antonin Chambolle (CEREMADE, CNRS & Université Paris-Dauphine PSL, Paris, France)

Korn and Poincaré-Korn inequalities for displacements with small jump set

Abstract: In this talk I will review recent rigidity results on so-called "$GSBD_p$" functions (Dal Maso, 2011) $u : \mathbb{R}^d \to \mathbb{R}^d$ ($d \geq 2$), which are functions with arbitrary jump set (rectifiable with finite $(d-1)$-dimensional measure) and a L^p ($p > 1$) control on a symmetrized gradient. The almost only available tool to control such functions is integration along lines. I will explain how one can show that up to a small set with finite perimeter, such functions coincide, where the jump is small, with Sobolev functions. This is based on joint works with F. Cagnetti, S. Conti, V. Crismale, G. Francfort, F. Iurlano, L. Scardia and extends results of Conti and Iurlano in dimension two (as well as variants of Friedrich and Solombrino).

Eleonora Cinti (Università di Bologna)

Some recent results in the study of fractional mean curvature flow

Abstract: We study a geometric flow driven by the fractional mean curvature (FMC). The notion of fractional mean curvature arises naturally when performing the first variation of the fractional perimeter functional. Our results concern, on one side, the formation of neckpinch singularities for the standard FMC flow and, on the other side, some a priori estimates and convergence to a sphere for the volume preserving case. This is a joint work with C. Sinestrari and E. Valdinoci.

Sara Daneri (Gran Sasso Science Institute)

Symmetry breaking and pattern formation for local/nonlocal interaction functionals

Abstract: In this talk I will review some recent results obtained in collaboration with E. Runa and A. Kerschbaum on the one-dimensionality of the minimizers of a family of continuous local/nonlocal interaction functionals in general dimension. Such functionals have a local term, typically the perimeter or its Modica-Mortola approximation, which penalizes interfaces, and a nonlocal term favouring oscillations which are high in frequency and in amplitude. The competition between the two terms is expected by experiments and simulations to give rise to periodic patterns at equilibrium. Functionals of this type are used to model pattern formation, either in material science or in biology. The difficulty in proving the emergence of such structures is due to the fact that the functionals are symmetric with respect to permutation of coordinates, while in more than one space dimensions minimizers are one-dimesnional, thus losing the symmetry property of the functionals. We will present new techniques and results showing that for two classes of functionals (used to model generalized anti-ferromagnetic systems, respectively colloidal suspensions), both in sharp interface and in diffuse interface models, minimizers are one-dimensional and periodic, in general dimension and also while imposing a nontrivial volume constraint.

Lucia De Luca (Istituto per le Applicazioni del Calcolo "M. Picone" IAC-CNR, Rome)

Topological singularities in periodic media

Abstract: We describe the emergence of topological singularities in periodic media within the Ginzburg-Landau model and the core-radius approach. The energy functionals of both models are denoted by $E_{\varepsilon,\delta}$, where ε represents the coherence length (in the Ginzburg-Landau model) or the core-radius size (in the core-radius approach) and δ denotes the periodicity scale. We carry out the

Γ-convergence analysis of $E_{\varepsilon,\delta}$ as $\varepsilon \to 0$ and $\delta = \delta_\varepsilon \to 0$ in the $|\log \varepsilon|$ energy regime, showing that the Γ-limit consists in the energy cost of finitely many vortex-like point singularities of integer degree. Such an energy cost depends on the parameter

$$\lambda := \min \left\{ 1, \lim_{\varepsilon \to 0} \frac{|\log \delta_\varepsilon|}{\varepsilon} \right\} ;$$

at scales larger than ε^λ the concentration process takes place "after" homogenization, whereas for scales less than ε^λ homogenization takes place "after" concentration.
The results I will present are obtained in collaboration with R. Alicandro, A. Braides, M. Cicalese, and A. Piatnitski.

Patrick Dondl (Albert Ludwig University of Freiburg)

Connected Coulomb Columns

Abstract: We first consider a notion of a *connected* perimeter, which is defined as the lower semicontinuous envelope of the perimeter of smooth approximating connected sets. From there, we develop a phase-field approximation of this connected perimeter based on the classical Modica-Mortola functional, where the connectedness constraint is introduced via a penalty functional. Using this connected perimeter, we study a version of Gamow's liquid drop model with a short range attractive perimeter-penalizing potential and a long-range Coulomb interaction of a uniformly charged mass in $\mathbb{R}^3$. Here we constrain ourselves to minimizing among the class of shapes that are columnar, i.e., constant in one spatial direction. In this setting, using the standard perimeter in the energy would lead to non-existence for any prescribed cross-sectional area due to the infinite mass in the constant spatial direction. Employing our connected perimeter, we instead obtain existence of minimizers for arbitrary mass as well as results regarding the shapes of minimizers in the small and large mass limit. We furthermore presesent numerical results – based on our aforementioned phase field approximation – for the intermediate mass regime.
This is joint work with Matteo Novaga (Pisa), Stephan Wojtowytsch (Princeton), and Steve Wolff-Vorbeck (Freiburg).

Selim Esedoglu (University of Michigan)

Variational extrapolation of numerical schemes for gradient flows

Abstract: A natural property to demand from discrete in time approximations to gradient flows is energy stability: Just like the exact evolution, the approximate evolution should decrease the cost function from one time step to the next. Often, approximation schemes that posess this desirable property, such as minimizing movements, are only first order accurate in time.
I will discuss a general (problem independent) procedure for boosting the order of accuracy of existing variational schemes for gradient flows, while preserving their desirable stability properties. The resulting high order versions are formulated only in terms of multiple calls of the original scheme per time step. Applications include threshold dynamics for motion by mean curvature, and the Jordan-Kinderlehrer-Otto scheme based on the 2-Wasserstein metric.

Julian Fischer (IST Austria)

Uniqueness and stability properties of multiphase mean curvature flow: An approach based on the variational (gradient flow) nature of the problem

Abstract: For many evolution problems for interfaces - like for instance multiphase mean curvature flow or the Mullins-Sekerka equation - appropriate weak solutions are known to exist globally in time, but the uniqueness of such weak solutions is either unknown or even known to fail in certain situations. At the same time, due to geometric singularities strong solution concepts are in general limited to local in time existence results. In the absence of a comparison principle, the relation between weak and strong solutions for interface evolution problems has remained a mostly open question.
We establish a weak-strong uniqueness principle for planar multiphase mean curvature flow: We prove that for curvature flow of planar networks, there is only a single BV solution prior to the first topology change. Our approach relies on the variational structure of mean curvature flow, being the gradient flow of the interface energy functional. For many minimization problems for interface energy functionals, the method of calibrations has made it possible to deduce the uniqueness of minimizers. Our approach to weak-strong uniqueness for mean curvature flow relies on a novel gradient-flow analogue of this concept of calibrations, basically allowing us to show that the route of steepest descent in the energy landscape is unique and stable with respect to perturbations.

Harald Garcke (University of Regensburg)

Phase field approaches to shape and topology optimization

Abstract: Variational problems that involve the minimization of a cost functional for shapes that can change their topological properties appear in many situations in mathematics and in applications. We introduce and analyse phase field methods to deal with such optimization problems. Problems to be addressed optimisation of elastic structures, spectral problems and additive manufacturing (3d printing). We show existence of solutions, derive first order conditions, study the sharp interface limit and show numerical computations.

Inwon Kim (UCLA)

Optimal transport with stopping times and Stefan problem

Abstract: We consider an optimal transport problem where the cost depends on the stopping time of Brownian motion from a given distribution to another. When the target measure is fixed, it is often called the optimal Skorokhod embedding problem in the literature, a popular topic in math finance. Under a monotonicity assumption on the cost, the optimal stopping time is given by the hitting time to a space-time barrier set. When the target measure is optimized under an upper bound constraint, we will show that the optimal barrier set leads us to the Stefan problem, a free boundary problem for the heat equation describing phase transition between water and ice. This is joint work (in progress) with Young-Heon Kim at UBC.

Xavier Lamy (Université Toulouse 3)

Saturn ring defect around a particle immersed in nematic liquid crystal

Abstract: Molecules in a nematic liquid crystal tend to align. In a joint work with Alama, Bronsard and Golovaty, we study the effect of a spherical particle which forces the molecules to align orthogonally

to its surface, thus enforcing a topological constraint on the alignment field. By minimizing the relevant energy among symmetric configurations, we obtain solutions whose singular set forms a Saturn ring around the particle: they approximate an $\mathbb{S}^2$-valued harmonic map which is singular at the equator of the spherical particle. Moreover, we are able to rule out the presence of point singularities, even though they would carry a negligible amount of energy.

Alice Marveggio (Institute for Science and Technology Austria)

On a non-isothermal Cahn-Hilliard model based on a microforce balance

Abstract: We consider a non-isothermal Cahn-Hilliard model based on a microforce balance. The model was derived by A. Miranville and G. Schimperna starting from the two fundamental laws of Thermodynamics, following M. Gurtin's two-scale approach. The main working assumptions are on the choice of the Ginzburg-Landau free energy, and on the behaviour of the heat flux as the absolute temperature tends to zero and to infinity. By deriving suitable a priori estimates and showing weak sequential stability of families of approximating solutions, we prove global-in-time existence for the initial-boundary value problem associated to the entropy formulation and, in a subcase, also to the weak formulation of the model. This is joint work with G. Schimperna, in Journal of Differential Equations. doi:10.1016/j.jde.2020.10.030 Preprint link: arXiv:2004.02618.

Simon Masnou (Université Claude Bernard Lyon 1)

Approximation of mean curvature flow: from phase field models to neural networks

Abstract: The first part of the talk will be devoted to a variant of the classical phase field approximation to mean curvature flow which promotes self-avoidance. Some theoretical and numerical issues will be discussed. In the second part of the talk, I will present an ongoing work on neural networks inspired by phase field models and able to learn the mean curvature flow. I will show a few numerical applications, including the approximation of solutions to the Steiner problem and the Plateau problem. It is a joint work with Elie Bretin (INSA Lyon), Roland Denis (CNRS), Chih-Kang Huang (UCBL), and Garry Terii (UCBL).

Christof Melcher (RWTH Aachen)

Emergent spin-orbit coupling in a spherical magnet

Abstract: We examine skyrmionic field configurations on a spherical magnet. Exploiting the Hamiltonian structure and concepts of angular momentum, we present a new family of localized solutions to the Landau–Lifshitz equation that are topologically distinct from the ground state and not equivariant. The approach illustrates emergent spin-orbit coupling arising from the loss of individual rotational invariance in spin and coordinate space — a common feature of condensed matter systems with topological phases.

Alpár R. Mészáros (Durham University, UK)

Weak solutions to the Muskat problem with surface tension via optimal transport

Abstract: Inspired by recent works on the threshold dynamics scheme for multi-phase mean curvature flow (by Esedoglu-Otto and Laux-Otto), we introduce a novel framework to approximate solutions of

the Muskat problem (describing the movement of two inviscid and incompressible fluids in a container driven by gravitational forces and the surface tension between them). Our approach is based on interpreting the Muskat problem as a gradient flow in a product Wasserstein space and in particular we construct weak solutions via a minimizing movements scheme. Rather than working directly with the singular surface tension force, we instead relax the perimeter functional with the heat content energy approximation of Esedoglu-Otto. The heat content energy allows us to show the convergence of the associated minimizing movement scheme in the Wasserstein space, and makes the scheme far more tractable for numerical simulations. Under a typical energy convergence assumption, we show that our scheme converges to weak solutions of the Muskat problem with surface tension. The talk is based on a recent joint work with Matt Jacobs and Inwon Kim (both from UCLA).

Connor Mooney (University of California, Irvine)

Singularity structures in solutions to the Monge–Ampère equation

Abstract: A celebrated theorem of Jörgens–Calabi–Pogorelov says that global convex solutions to the Monge–Ampère equation $\det(D^2 u) = 1$ are quadratic polynomials. On the other hand, an example of Pogorelov shows that local solutions can have line singularities. It is natural to ask which singular structures can appear in functions that solve the Monge–Ampère equation outside of a small set. We will discuss examples of functions that solve the equation away from finitely many points, but exhibit polyhedral and Y-shaped singularities. We will discuss geometric and applied motivations for constructing such examples, as well as their connection to an obstacle problem for the Monge–Ampère equation.

Matteo Novaga (University of Pisa)

Variational models for charged liquid drops

Abstract: I will discuss some variational models for charged liquid drops, where the interaction energy between the charged particles is given by a repulsive potential. Depending on such potential, I will discuss existence or nonexistence and qualitative properties of minimizers.

Alessandra Pluda (Università di Pisa)

Resolution of singularities of the network flow

Abstract: The curve shortening flow is an evolution equation in which a curve moves with normal velocity equal to its curvature (at any point and time) and can be interpreted as the gradient flow of the length. We consider the same flow for networks (finite unions of sufficiently smooth curves whose end points meet at junctions). Because of the variational nature of the problem, one expects that for almost all the times the evolving network will possess only triple junctions where the unit tangent vectors forms angles of 120 degrees (regular junctions). However, even if the initial network has only regular junctions, this property is not preserved by the flow and junctions of four or more curves may appear during the evolution. The aim of this talk is first to describe the process of singularity formation and then to explain the resolution of such singularities and how to continue the flow in a classical PDE framework.
This is a research in collaboration with Jorge Lira (Universidade Federal do Ceará), Rafe Mazzeo (Stanford University) and Mariel Saez (Pontificia Universidad Católica de Chile).

Marcello Ponsiglione (Sapienza University of Rome)

Stability results for nonlocal geometric evolutions

Abstract: We introduce a notion of uniform convergence for local and nonlocal curvatures and we propose an abstract method to prove the convergence of the corresponding geometric flows, within the level set formulation. We apply such a general theory to characterize the limits of several nonlocal geometric evolutions. We study the limit of the s-fractional mean curvature flows as $s \to 0^+$ and $s \to 1^-$. Moreover, in analogy with s-fractional mean curvature flows, we introduce the notion of s-Riesz curvature flows and characterize its limit as $s \to 0^-$. Furthermore, using a suitable core-radius regularization, we define s-fractional perimeters and s-fractional curvatures also for $s \geq 1$ and we show that - as the core-radius tends to 0 - the corresponding geometric flows converge to the classical mean curvature flow. Eventually, we discuss the limit behavior as $r \to 0^+$ of the flow generated by a regularization of the r-Minkowski content. The results discussed here are obtained in collaboration with A. Cesaroni (Padova), L. De Luca (CNR), A. Kubin (Roma "La Sapienza"), M. Novaga (Pisa).

Tobias Ried (Max Planck Institute for Mathematics in the Sciences)

A variational approach to the regularity of optimal transportation

Abstract: In this talk I want to present a purely variational approach to the regularity theory for the Monge–Ampère equation, or rather optimal transportation, introduced by Goldman–Otto. Following De Giorgi's strategy for the regularity theory of minimal surfaces, it is based on the approximation of the displacement by a harmonic gradient, which leads to a one-step improvement lemma, and feeds into a Campanato iteration on the $C^{1,\alpha}$-level for the displacement. We extend the result of Goldman–Otto for the Euclidean cost function to the case of general cost functions. One of the new contributions is the use of almost-minimality: if the cost is quantitatively close to the Euclidean cost function, a minimizer for the optimal transport problem with general cost is an almost-minimizer for the one with quadratic cost. This allows us to reprove the $C^{1,\alpha}$-regularity result of De Philippis–Figalli, bypassing Caffarelli's celebrated theory.
This is joint work with F. Otto and M. Prod'homme.

Matthias Röger (TU Dortmund)

A Willmore approximation arising from an Allen–Cahn/Cahn–Hilliard system

Abstract: We consider a diffuse interface evolution accounting for multiple microscopic mechanism that was proposed by Karalis and Katsoulakis (2007). The associated dissipation functional formally approximates the Willmore functional. In this talk we present a corresponding Gamma convergence result.

Giorgio Saracco (Scuola Internazionale Superiore di Studi Avanzati Trieste)

Convexity properties of the isoperimetric profile

Abstract: Given an open, bounded set Ω we consider the isoperimetric profile $\mathcal{J}$ that to each volume $V \in [0, |\Omega|]$ associates the least perimeter $P(E)$ among Borel subsets E of Ω needed to enclose the given volume. We shall prove that for a wide class of planar sets, which encompasses convex sets, there exists a threshold $\bar{V}$ such that $\mathcal{J}$ is concave below it and convex above it. Moreover, $\mathcal{J}^2$ is globally convex.

In order to prove these properties, a full characterization of the isoperimetric sets will be provided. Some comments on the n-dimensional case will be given.
Based on joint works with G.P. Leonardi (Università di Trento)

Theresa Simon (University of Bonn)

Skyrmions and stability of degree 1 harmonic maps from the plane to the two-dimensional sphere

Abstract: Skyrmions are topologically nontrivial patterns in the magnetization of extremely thin ferromagnets. Typically thought of as stabilized by the so-called Dzyaloshinskii-Moriya interaction (DMI), or antisymmetric exchange interaction, arising in such materials, they are of great interest in the physics community due to possible applications in memory devices.
In this talk, we will characterize skyrmions as local minimizers of a two-dimensional limit of the full micromagnetic energy, augmented by DMI and retaining the nonlocal character of the stray field energy. In the regime of dominating Dirichlet energy, we will provide rigorous predictions for their size and "wall angles". The main tool is a quantitative stability result for harmonic maps of degree 1 from the plane to the two-dimensional sphere, relating the energy excess of any competitor to the homogeneous H-distance to the closest harmonic map.

Raghav Venkatraman (Carnegie Mellon University and Courant Institute of Mathematical Sciences)

Highly Anisotropic Nematic Liquid Crystals

Abstract: We report on recent progress on nematic liquid crystals with disparate elastic constants. This work is inspired by lab experiments on nematic to isotropic phase transitions in liquid crystals that display characteristic features such as cusp and corner singularities along the nematic isotropic phase boundary. We will present results that explain such phenomena.
This is joint work with Dmitry Golovaty (Akron), Michael Novack (Austin), and Peter Sternberg (Indiana).

Aaron Nung Kwan Yip (Purdue University)

Stability of Self-Similar Solutions of Surface Diffusion

Abstract: Surface diffusion (SD) is a curvature driven flow where a (hyper-)surface evolves by the surface Laplacian of its mean curvature. It is a fourth order parabolic equation. Compared with its second order counterpart, motion by mean curvature (MMC), for which maximum principle is available, much less is known for SD. I will present a stability result for self-similar solutions. Though the approach is based on linearization, and it only works for the evolution of graphs, it is quite robust and works for both MMC and SD. I will also present an attempt to analyze the pinch-off phenomena for axisymmetric surfaces. The former is based on a joint work with Hengrong Du and the latter with Gavin Glenn.

Konstantinos Zemas (University of Münster)

Rigidity estimates for isometric and conformal maps on the sphere

Abstract: In this talk I would like to present the results obtained in collaboration with S. Luckhaus on both linear and nonlinear stability aspects of the class of rigid motions (resp. Möbius transformations) of the standard round sphere among maps from the sphere into the ambient Euclidean space.
Unlike similar in flavor results for maps defined on domains, not only an isometric (resp. conformal) deficit is necessary in this more flexible setting, but also a deficit measuring the distortion of the sphere under the maps in consideration. The latter is defined as an associated isoperimetric type of deficit.
The focus will mostly be on the case when the ambient dimension is 3 and we also explain why, in both cases, the estimates are optimal in their corresponding settings. The adaptations needed in higher dimensions will also be addressed. We also obtain linear stability estimates for both cases in all dimensions. These can be regarded as Korn-type inequalities for the combination of the quadratic form associated with the isometric (resp. conformal) deficit on the sphere and the isoperimetric one.

Caterina Ida Zeppieri (University of Münster)

Stochastic homogenisation of free-discontinuity problems

Abstract: In this talk I will present a stochastic homogenization result for free-discontinuity functionals. Assuming stationarity for the random volume and surface integrands, we prove the existence of a homogenized random free-discontinuity functional, which is deterministic in the ergodic case. Moreover, by establishing a connection between the deterministic convergence of the functionals at any fixed realization and the pointwise Subadditive Ergodic Theorem by Akcoglou and Krengel, we characterize the limit volume and surface integrands in terms of asymptotic cell formulas. Our homogenization result extends to the SBV-setting the classical results by Papanicolaou and Varadhan, Kozlov, and Dal Maso and Modica, formulated in the Sobolev setting. Recent developments obtained in the BV setting and for degenerate integrands will be also addressed. Joint work with F. Cagnetti (Sussex) , G. Dal Maso (SISSA), and L. Scardia (Heriot-Watt)
