Workshop

"Calculus of Variations: a New Generation"

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organized by
Lorenzo Portinale, Simone Di Marino, Emanuela Radici

Abstracts

Adolfo Arroyo-Rabasa (Université Catholique de Louvain)

Local Poincaré constants and BMO functionals for BV functions

Abstract: Since the characterizing of constant maps lacking a classical derivative (Bourgain et al.), there has been a growing interest on the link between derivatives and limits of BMO seminorms. Ambrosio et al. described perimeter using mean oscillations, later extended to describe the total variation of SBV functions as limits of BMO-type seminorms on ε-size cubes (as $\varepsilon \to 0$). However, this uniform ε-size constraint hinders a precise description for general BV functions. Can we relax this model to find a BMO-type description of BV functions?
In this talk, I will give an answer to this question. First, I will introduce the concept of local Poincaré constant (LPC) for a BV function, as a tool to understand the relation between its mean oscillation and its total variation at small scales. Along with this new quantity, I will introduce a geometric relaxation of the functionals by Ambrosio et al., by considering cubes of sidelength smaller than or equal to ε. I will then explain how our functionals converge (as $\varepsilon \to 0$) to a functional defined on BV, represented by integration in terms of the LPC and the total variation.
Additionally, I will discuss a cell-formula representation for the LPC, as the maximum mean oscillation amongst BV blow-ups. As we shall see, this demonstrates our new functional extends the original one from SBV to all BV functions. If time permits, I will also share with you an interesting conjecture concerning a more general representation of our functional, connected with our understaning of the LPC, the fine properties of BV functions, and covering theorems for gradient measures.
This is joint work with Paolo Bonicatto (Turin) and Giacomo del Nin (MPI Leipzig)

Annika Bach (TU/e, Eindhoven)

Stacking faults in the limit of a discrete model for partial edge dislocations

Abstract: In this talk we introduce a simple next-to-nearest neighbour lattice model for partial edge dislocations and stacking faults and we address its variational coarse graining. In the framework of Γ-convergence we show that as the lattice spacing tends to zero suitable scaled versions of the model converge to a continuum mechanical model accounting for partial dislocations and stacking faults. The limiting energy consists of three contributions, a core contribution centred around each partial dislocation, a Coulombian interaction energy between the latter, and a surface energy concentrated on the stacking faults.

This is joint work with M. Cicalese (TU Munich), A. Garroni (Sapienza University of Rome), and G. Orlando (Politecnico di Bari).

Giulia Bevilacqua (University of Pisa)

Classical solutions for a soap film capillarity problem

Abstract: In this talk, we study regularity of the soap film capillarity problem, that is soap films are chosen to be sets of finite perimeter containing an assigned amount of volume and satisfying a topological spanning condition. For plane boundaries, we show that these minimizers are normal smooth graphs with positive constant mean curvature and meeting orthogonally the boundary. This is a joint work with S. Stuvard and B. Velichkov.

Elise Bonhomme (Université Libre de Bruxelles)

Variational methods applied to discrete models in brittle damage

Abstract: In this talk, I will study a discrete model of brittle damage in different regimes where the damaged zone concentrates on vanishingly small sets. We will identify the nature of the effective limit models obtained by means of an asymptotic analysis based on the Γ-convergence of the total energies. I will recall the mechanical model of brittle damage introduced by Francfort and Marigo [5], specified to the discrete setting where the total energies are restricted to piecewise affine continuous displacements. More precisely, given ε and $\eta_\varepsilon > 0$, we consider a linearly elastic material, whose reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$, which is composed of only two phases: a damaged phase (where the elasticity of the medium is altered) and a sound one, whose elasticity properties are given by η_ε and 1 respectively. Introducing the characteristic function of the damaged region, $\chi \in L^\infty(\Omega; \{0,1\})$, Francfort-Marigo's model consists in defining the total energy associated to a displacement $u \in H^1(\Omega; \mathbb{R}^2)$ as the sum of the elastic energy stored inside the material and a dissipative energy, taken as proportional to the volume of the damaged zone:

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega (\chi\eta_\varepsilon + (1-\chi))\,|e(u)|^2\,dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi\,dx,$$

where $e(u) = \left(\nabla u + \nabla u^T\right)/2$ is the linearized elastic strain and $\kappa/\varepsilon > 0$ is the material's toughness in the damaged regions. Note that the elasticity coefficients η_ε of the weak material degenerate while the diverging character of κ/ε as $\varepsilon \searrow 0$ forces the damaged zones to concentrate on vanishingly small sets. Here, we consider the total energies restricted to couples $(u,\chi) \in \mathcal{C}^0(\Omega;\mathbb{R}^2) \times L^\infty(\Omega,\{0,1\})$ in the finite element set

$$(u,\chi) \in X_{h_\varepsilon}(\Omega),$$

for which there exists a triangulation $\mathbf{T}_{h_\varepsilon}$ of Ω, whose mesh-size is of order $h_\varepsilon > 0$, such that u is affine and χ is constant on each of its triangle. According to the convergence rates $\alpha = \lim_{\varepsilon\searrow 0} \frac{\eta_\varepsilon}{\varepsilon} \in [0,+\infty]$ and $\beta = \lim_{\varepsilon\searrow 0} \frac{h_\varepsilon}{\varepsilon} \in [0,+\infty]$, we will exhibit the following effective limit models (some of them are still in progress):

Regime	**Effective limit model**
$\alpha = +\infty$ or $\beta = +\infty$	linear elasticity (see [3])
$\alpha = \beta = 0$	trivial model (see [3])
$\alpha = 0$ and $\beta \in (0,+\infty)$	brittle fracture (see [2])
$\alpha \in (0,+\infty)$ and $\beta = 0$	Hencky plasticity
$\alpha, \beta \in (0,+\infty)$	in between plasticity and brittle fracture (see [1, 4])

References

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[3] Babadjan J.F., Iurlano F. & Rindler F., *Concentration versus oscillation effects in brittle damage*, Comm. Pure Appl. Math., **74** (2021) 1803–1854

[4] Focardi M. & Iurlano F., *Asymptotic analysis of Ambrosio-Tortorelli energies in linearized elasticity*, SIAM J. Math. Anal., **46** (2014) 2936–2955

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Camilla Brizzi (TUM, München)

p-Wasserstein barycenters

Abstract: The talk will be about the barycenters of N probability measures with respect to the p-Wasserstein metric ($p > 1$), which generalizes the notion of Wasserstein barycenters for $p = 2$, introduced by Agueh and Carlier. In particular it will be showed that

- p-Wasserstein barycenters of absolutely continuous functions are unique and again absolutely continuous;
- p-Wasserstein barycenters admit a multi-marginal OT formulation;
- the optimal multi-marginal plan is unique and of Monge form if the marginals are absolutely continuous and it has an explicit parametrization as a graph over any marginal space.

A key ingredient is a quantitative injectivity estimate for the (highly non-injective) p-barycenter map on the support of an optimal multi-marginal plan. Some examples in one dimension will be also discussed. This is a joint work with G. Friesecke and T. Ried.

Leon Bungert (University of Würzburg)

An adversarial mean curvature flow

Abstract: In this talk I will draw a connection between adversarial robustness in machine learning and mean curvature flow. The connection arises in the context of adversarial training which is a robust optimization method for solving classification problems. Relying on a perspective from a joint work with N. García Trillos and R. Murray that recasts adversarial training as a regularization problem, I will introduce a modified training method that constitutes a minimizing movements scheme for a nonlocal perimeter functional. Since the scheme is monotone and consistent as the adversarial budget vanishes, the scheme approximates a weighted mean curvature flow. This analysis requires a variety of tools for working with the subdifferential of a supremal-type nonlocal total variation and its regularity properties. This talk is based on joint work with Tim Laux and Kerrek Stinson.

Sara Daneri (Gran Sasso Science Institute, L'Aquila)

Continuous symmetry breaking: a rigorous approach

Abstract: At the base of spontaneous pattern formation is universally believed to be the competition between short range attractive and long range repulsive forces. Though such a phenomenon is observed in experiments and simulations, a rigorous understanding of the mechanisms at its base is still in most physical problems a challenging open problem. The main difficulties are due to the nonlocality of the interactions and, in more than one space dimensions, the symmetry breaking phenomenon (namely the fact that the interactions have a larger group of symmetries than that of their minimizers).
In this talk we consider a general class of isotropic functionals in dimension $d \geq 2$, typical in physical models, in which a surface term favouring pure phases competes with a nonlocal term with power law kernel favouring alternation between different phases. Close to the critical regime in which the two terms are of the same order, we give a rigorous proof of the conjectured symmetry breaking and pattern formation for global minimizers, in the shape of domains with flat boundary (e.g. stripes or lamellae).
The natural framework in which our approach is set and developed is the one of calculus of variations and geometric measure theory. Among others, our approach relies on detecting a nonlocal curvature-type quantity which is controlled by the energy functional and whose finiteness implies flatness for sufficiently regular boundaries.
This is a joint work with E. Runa.

Cristiana De Filippis (University of Parma)

Nonlinear potentials at the fractional scale: sharp regularity

Abstract: Schauder estimates are a basic tool in elliptic and parabolic PDE and ultimately establish that solutions are as regular as coefficients. They intervene in many situations, such as higher regularity of solutions to problems showing any kind of ellipticity, including free boundaries, bootstrap processes, existence theorems and so on. In the linear case they are a classical topic, with results obtained since the '20s of the past century by Hopf, Giraud, Caccioppoli and Schauder. New proofs were achieved over the years by Campanato (via proper function spaces), Trudinger (via convolution methods), Leon Simon (via blow-up). Nonlinear versions were settled by Giaquinta & Giusti, Ivert, DiBenedetto, Manfredi. All these results deal with uniformly elliptic operators, and unavoidably rely on perturbation methods, i.e., freezing coefficients and comparing original solutions to solutions with problems without coefficients. Such methods do not any longer deliver results in nonuniformly elliptic problems, for which homogeneous a priori estimates are lost and standard iteration arguments break down. We shall present a solution to the longstanding problem of establishing the validity of Schauder estimates in the nonlinear, nonuniformly elliptic setting, under the sharp growth rate of the ellipticity ratio. This features a novel approach to a priori gradient bounds that does not rely on perturbation although the problems involved are non-differentiable. From recent, joint work with Giuseppe Mingione (Parma).

Annette Dumas (University Claude Bernard - Lyon 1)

Lipschitz regularity of the trajectories minimizing the total variation in a congested setting

Abstract: The problem I will present is motivated by the study of a Mean Field Game model whose theory was simultaneously introduced by Lasry and Lions and by Caines, Huang and Malhamé in 2006. The model consists in studying a population in a city where each agent jumps to move from

one place to another. Each inhabitant minimizes a cost composed of the number of jumps and an increasing function of the density of the population. The solution to this problem is a probability measure on the trajectories which is a Nash equilibrium.
The probability measure Q on the trajectories can be seen as a trajectory of the density of the population which leads us to the minimization of a variational problem which depends on the L^1-norm of the speed of the density which is linked to the number of jumps and the additional cost which is associated with the increasing function of the density. Density constraints are also added to the problem. We will see that the solution exists, is unique and is Lipschitz in time, despite the discontinuous trajectories taken by each agent. With additional hypothesis on the data, boundedness or continuity in space can be obtained with Dirichlet conditions in time.
The aspect of the solutions are given by the Euler-Lagrange equations which show that in space, either the solution is constant, or it follows the critical points of the cost. Numerical simulations are carried out on a simple example by using the fast dual proximal gradient method from Beck which validates the theoretical framework. It is possible to apply this model to multi-species modelling.

Katharina Eichinger (École Polytechnique, Paris)

On the turnpike property in mean field games

Abstract: In this talk we prove the exponential turnpike property for a class of mean field games on $\mathbb{R}^d$. The exponential turnpike property states that optimal trajectories stay exponentially close to a stationary state, called turnpike, if they are far from the initial and final time. Our technique is based on coupling by reflection adapted to controlled processes allowing us to treat controlled dynamics governed by an asymptotically convex potential. This enables us to prove existence and uniqueness of mean field game problems and their ergodic counterpart without monotonicity assumptions on the cost but rather a smallness condition on the dependence of the measure variable, and finally the exponential turnpike property. Based on joint work with Alekos Cecchin, Giovanni Conforti and Alain Durmus.

Xavier Fernandez-Real (EPFL, Lausanne)

On the one-phase free boundary problem

Abstract: In this talk, we will introduce the one-phase free boundary problem and present some recent developments.

André Guerra (Institute for Theoretical Studies, ETH, Zürich)

Harmonic maps and the vectorial obstacle problem: singularities vs free boundaries

Abstract: I will discuss some recent results obtained in collaboration with A. Figalli, S. Kim and H. Shahgholian. We consider minimizers of the Dirichlet energy among maps constrained to take values outside a smooth domain O in $\mathbb{R}^m$. These minimizers can be thought of as solutions of a vectorial obstacle problem, or as harmonic maps into the manifold-with-boundary given by the complement of O. I will discuss results concerning the regularity of the minimizers, the location of their singularities, and the structure of the free boundary.

Jasper Hoeksema (TU/e, Eindhoven)

Gradient flow formulations for singular jump processes in metric spaces

Abstract: In this talk we discuss a generalisation of variational formulations for the flow corresponding to jump processes with a singular kernel, to the setting of metric spaces. Similar to the local case, this problem is well-posed if the densities of the laws with respect to a reversible invariant measure are bounded from above and above, and a certain Lipschitz density result holds. We will give various examples and discuss cases where this type of approach fails.
Joint work with Oliver Tse and Riccarda Rossi.

Domenico Angelo La Manna (University Federico II, Naples)

Fractional heat equation and sets of finite perimeter

Abstract: In this talk we present a result about characterization of sets of finite local and non local perimeter via Γ-convergence. As an application we give a short proof of the isoperimetric inequality, both in the local and in the non local case. The results are obtained in collaboration with Andrea Kubin.

Tim Laux (University of Regensburg)

Energy convergence of the Allen-Cahn equation for mean convex mean curvature flow

Abstract: In this talk, I'll present a work in progress in which I positively answer a question of Ilmanen (JDG 1993) on the strong convergence of the Allen-Cahn equation to mean curvature flow when starting from well-prepared initial data around a mean convex surface. As a corollary, the conditional convergence result with Simon (CPAM 2018) becomes unconditional in the mean convex case.

Alice Marveggio (HCM, Bonn)

Stability of multiphase mean curvature flow beyond a circular topology change

Abstract: The evolution of a network of interfaces by mean curvature flow features the occurrence of topology changes and geometric singularities. As a consequence, classical solution concepts for mean curvature flow are in general limited to a finite time horizon. At the same time, the evolution beyond topology changes can be described only in the framework of weak solution concepts (e.g., Brakke solutions), whose uniqueness may fail. Following the relative energy approach, we prove a quantitative stability estimate holding up to the singular time at which a circular closed curve shrinks to a point. This implies a weak-strong uniqueness principle for weak varifold-BV solutions to planar multiphase mean curvature flow beyond circular topology changes. We expect our method to have further applications to other types of shrinkers.
This talk is based on a joint work with Julian Fischer, Sebastian Hensel and Maximilian Moser.

Roberta Marziani (University of L'Aquila)

Γ-convergence for plane to wrinkles transition problem

Abstract: Motivated by physical experiments, we consider a variational problem modelling transition between flat and wrinkled region in a thin elastic sheet. We then identify the Γ-limit as the sheet thickness goes to 0. The limiting problem is scalar and convex, but constrained and posed for measures. For the Γ-liminf inequality we first pass to quadratic variables so that the constraint becomes linear, and then obtain the lower bound by Reshetnyak's theorem. The construction of the recovery sequence for the Γ-limsup inequality relies on mollification of quadratic variables, and careful choice of multiple construction parameters. Eventually for the limiting problem we show existence of a minimizer and equipartition of the energy for each frequency. This is based on a joint work with Peter Bella (TU Dortmund).

Simone Rademacher (LMU, München)

The effective mass problem for the classical polaron

Abstract: The polaron is a model for an electron moving through an ionic crystal. Due to the interaction of the electron with its self-induced polarization field, the electron slows down and its effective mass increases. Landau and Pekar conjectured an asymptotic formula for the effective mass whose mathematical verification is an outstanding and challenging problem in mathematical physics. Landau and Pekar's conjecture is based on heuristics on traveling wave solutions for the Landau-Pekar equations, known to be a classical approximation for the quantum polaron. In this talk we first consider regularized Landau-Pekar equations, show existence of traveling wave solutions and, based on this, provide a rigorous definition of the effective mass. However for the non-regularized equations, traveling waves are conjectured to not exist. We provide a novel approach for the definition of the effective mass based on a variational principle minimizing the energy functional over states with given initial velocity. The resulting formula for the effective mass agrees with the famous conjecture by Landau and Pekar. This is based on joint work with Dario Feliciangeli and Robert Seiringer.

Bozhidar Velichkov (University of Pisa)

Boundary regularity for optimal partition

Abstract: This talk is based on a joint work with Roberto Ognibene on the problem of optimal partition of a fixed domain D with respect to the sum of the principal eigenvalues. The regularity of the free interfaces arising in this type of free boundary problems were extensively studied in the interior of the domain D in relation with harmonic maps (Caffarelli-Lin, Gromov-Schoen) and segregation problems (Conti-Terracini-Verzini). Recently with Roberto Ognibene we proved regularity results for the free interface up to fixed boundary; in particular, we showed that the subset of points of minimal frequency is regular and that, at these points, the interior free interface is approaching the boundary orthogonally in a smooth way.
