HIM Trimester 2021 report
Harmonic Analysis and Analytic Number Theory
Dual Trimester Program

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1 Opening Day

The program took off with two online talks on May 3, 2021: one by Larry Guth (MIT) and one by Terence Tao (UCLA). Each was attended by 90-100 people.

- Larry Guth (MIT): Sharp examples for decoupling and related questions
  Abstract: This talk will be about some open questions related to decoupling. The starting question will be, what are the known sharp examples for decoupling? This question can be made precise in various ways. Depending on how exactly we phrase the question, there can be many sharp examples or very few known sharp examples. For some versions of the question, the known sharp examples are very rare and are based on number theoretic structure. Is it true that all nearly sharp examples have some sort of number theoretic structure? I have no idea. In the talk, we will make this question precise, see how it connects to some problems in harmonic analysis, and describe an obstacle to understanding it using current harmonic analysis techniques.

- Terence Tao (UCLA): Pseudorandomness of the Liouville function
  Abstract: The Liouville pseudorandomness principle (a close cousin of the Mobius pseudorandomness principle) asserts that the Liouville function $\lambda(n)$, which is the completely multiplicative function that equals 1 at every prime, should be "pseudorandom" in the sense that it behaves statistically like a random function taking values in $-1, +1$. Various formalizations of this principle include the Chowla conjecture, the Sarnak conjecture, the local uniformity conjecture, and the Riemann hypothesis. In this talk we survey some recent progress on some of these conjectures.

2 Polynomial Methods Summer School

Powerful progress on a wide selection of problems spanning across harmonic analysis and number theory has involved methods using auxiliary polynomials. On the more number-theoretic side, this includes the recent resolution of the dimension growth conjecture via the determinant method, as well as Stepanov's method for proving square-root cancelation of exponential sums, and results in transcendence theory related to open conjectures in algebraic number theory. On the more analytic side, applications of polynomial methods include the resolution of the Kakeya conjecture over finite fields, progress in incidence geometry, and new methods for tackling difficult questions on restriction inequalities.This summer school gathered together all these "polynomial methods" with accessible lecture series from each of these perspectives. Graduate students and postdocs gained intuition and technical skills for how these methods can be applied in a wide range of settings. While the polynomial methods presented in this summer school may have "evolved" independently, the lecture series explored connections and parallels that unify the methods. Attendance varied but averaged around 40-100 viewers (online).

Lecture series were given by Valentin Blomer, Samit Dasgupta, Roger Heath-Brown, Marina Iliopoulou, Hong Wang. Descriptions of these lectures follow:

- Valentin Blomer (University of Bonn) The polynomial method for point counting and exponential sums
  Abstract: We show how families of auxiliary polynomials can be used to count the number of points on certain types of curves over finite fields and to estimate exponential sums and character sums.

- Samit Dasgupta (Duke University) An introduction to auxiliary polynomials in transcendence theory
  Abstract: Broadly speaking, transcendence theory is the study of the rationality or algebraicity properties of quantities of arithmetic or analytic interest. For example, Hilbert's 7th problem asked "Is a^b always transcendental if $a \neq 0, 1$ is algebraic and b is irrational algebraic?" An affirmative answer was proven by Gelfond and Schneider in 1934-35. In the late 1960s, Baker generalized this result in spectacular fashion, proving that if a set of logarithms of algebraic numbers is linearly dependent over the algebraic numbers, then it is in fact linearly dependent over the rational numbers. All of these results used the technique of auxiliary polynomials. To be precise, Baker showed that if a set of logarithms of algebraic numbers is linearly dependent over the algebraic numbers, then there exists a polynomial whose vanishing set is of a particular form; the existence of this polynomial is then used to deduce the linear dependence of the original logarithms over the rational numbers. We will begin the course by proving Baker's Theorem. Next, we will describe the structural rank conjecture, which predicts the rank of a matrix of logarithms of algebraic numbers. We will prove the beautiful theorem of Masser and Waldschmidt that provides a lower bound on this rank, giving a partial result toward the struc-

tural rank conjecture. Waldschmidt's theorem gives the construction of a certain auxiliary polynomial, and Masser's theorem provides the desired lower bound given the existence of this auxiliary polynomial. Techniques used in the course will include complex analysis and commutative algebra.

- Roger Heath-Brown (University of Oxford) The Determinant Method
  Abstract: Lecture 1 will set the background for the course, describing the problem of counting rational points on algebraic varieties, the phenomena that can arise, and some of the results which have been proved. Lecture 2 will prove the basic theorem of the p-adic Determinant Method. Lecture 3 will describe variants of the determinant method, and will use the Approximate Determinant Method to examine consecutive "powerful" numbers.

- Marina Iliopoulou (University of Kent) Three polynomial methods for point counting
  Abstract: During these lectures, we will describe (a) the polynomial method that Dvir developed to solve the Kakeya problem in finite fields, (b) polynomial partitioning, developed by Guth and Katz to solve the Erdos distinct distances problem in the plane, and (c) the slice rank method, developed by Croot, Lev, Pach/ Ellenberg, Gijswijt to show that, in finite-field settings, sets with no 3-term arithmetic progressions are small.

- Hong Wang (Institute for Advanced Study) The restriction problem and the polynomial method
  Abstract: Stein's restriction conjecture is about estimating functions with Fourier transform supported on a hypersurface, for example, a sphere in $\mathbb{R}^n$. These functions can be decomposed into a sum over wave packets supported on long thin tubes. Guth introduced the polynomial method in restriction theory, in particular, to study the intersection of those tubes. In this mini-course, we give a brief introduction of Stein's restriction conjecture and the Kakeya conjecture. Then understand how the polynomial method is used to study these problems

3 Circle Method Summer School

The Circle Method emerged one hundred years ago from ideas of Ramanujan, Hardy and Littlewood, and quickly became the most powerful analytic method for counting solutions to Diophantine equations. As the Circle Method enters its second century, new work is making significant advances both in strengthening results in classical Diophantine settings, and in demonstrating applications in novel settings. This includes function field, number field, adelic, geometric, and harmonic analytic applications, with striking consequences in areas such as ergodic theory, subconvexity for L-functions, and the Langlands program. This summer school for graduate students and postdocs presented accessible lecture series that demonstrated how to apply the Circle Method in a wide variety of settings. Participant gained both a foundational understanding of the core principles of the Circle Method, and an overview of cutting-edge applications of the method. Attendance varied but averaged around 40-100 viewers (online).

Lecture series were given by: Timothy Browning (IST Austria), Jayce Robert Getz (Duke University), Yu-Ru Liu (University of Waterloo), Ritabrata Munshi (Indian Statistical Institute), Simon Myerson (University of Warwick), Lillian B. Pierce (Duke University). Additional research seminars were given by: Kirsti Biggs (University of Gothenburg), Julia Brandes (University of Gothenburg), Oscar Marmon (Lund University), Damaris Schindler (University of Göttingen), Pankaj Vishe (Durham University). Descriptions of the lecture series and research seminars follow:

- Kirsti Biggs (University of Gothenburg) Ellipsephic applications of the circle method
  Abstract: Ellipsephic sets are subsets of the natural numbers defined by digital restrictions in a given base — such sets have a fracta$L$-like structure which can be seen as a p-adic analogue of generalised real Cantor sets. The recent work of Maynard on primes with missing digits can be seen as an ellipsephic problem, although in this talk we focus on smaller sets of permitted digits, one motivating example being the set of natural numbers whose digits are squares. I will discuss applications of the circle method to Diophantine problems involving ellipsephic sets, and highlight the key features of such results.

- Julia Brandes (Universit of Gothenburg) Simultaneous diagonal and non-diagonal equations
  Abstract: The circle method can be used to effectively count solutions to diagonal equations. It can also be used (though less effectively) to count solutions to equations of general shape. Can we do both at once, without abandoning the special shape of the diagonal equations? We will outline a strategy to address this problem by combining approaches in the style of Birch's theorem with those stemming from systems of diagonal equations.

- Timothy Browning (IST Austria) The circle method over function fields and applications to geometry (series)
  Abstract: The solubility of cubic forms over arithmetically interesting fields has long been at the centre of number theory. I will briefly survey what is known for finite fields, local fields and global fields. The main goal is then to introduce the mechanics of the Hardy-Littlewood circle method over the function field $\mathbb{F}_q(t)$ through the prism of cubic forms. After introducing the basics, I will explain how to deal with $\mathbb{F}_q(t)$-solubility for diagonal cubic forms that are defined over the finite field $\mathbb{F}_q$. It turns out that these techniques can also shed light on the geometry of the moduli space parameterising rational curves on varieties. I will illustrate this by showing how the $\mathbb{F}_q(t)$-version of the circle method can recover classical facts about the smoothness/dimension/irreducibility of the Fano variety of lines on diagonal cubic hypersurfaces.

- Jayce Getz (Duke University) New avenues for the circle method (series)
  Abstract: Motivated by research arising from automorphic representation theory, I will present some ideas that should open up new avenues of research in the circle method. In the first half of the lectures I will discuss an adelic version of the delta-method of Duke, Friedlander, Iwaniec and Heath-Brown and state a (mostly conjectural) nonabelian analogue that I believe warrants further study. In the second half of the lectures I will discuss Poisson summation formulae and Fourier transforms for special families of varieties including, for example, the zero locus of a quadratic form. My hope is that they will allow the standard techniques of analytic number theory that rely on Fourier theory on a vector space to be broadly generalized.

- Yu-Ru Liu (University of Waterloo) An introduction to the circle method (series)
  Abstract: This course will focus on the basic principles of the circle method. We will start with Waring's problem about representations of positive integers as a sum of fixed powers. Then we will study Vinogradov's mean value theorem about a system of equal sums of powers. If time permits, we will consider extensions of these problems to the multidimensional setting and the function field.

- Oscar Marmon (Lund University) Rational points on quartic hypersurfaces
  Abstract: By work of Heath-Brown and Hooley, it is known that the Hasse principle holds for non-singular cubic forms in at least nine variables. The situation for forms of higher degree is much less satisfactory. Browning and Heath-Brown established the Hasse principle for non-singular quartic forms in at least 41 variables, and Hanselmann subsequently showed that 40 variables suffice. In joint work with Pankaj Vishe, we have been able to drastically reduce the number of variables needed to 30. To obtain the improvement, we combine Heath-Brown's delta-symbol version of the circle method with a van der Corput differencing technique.

- Ritabrata Munshi (Indian Statistical Institute) The circle method and the analytic theory of L-functions (series)
  Abstract: The aim of this series of talks will be to introduce variants of the delta method and to show how they can be employed to tackle various problems in the analytic theory of automorphic forms and L-functions. We will start by applying the classical Kloosterman's circle method and/or the delta method of Duke, Friedlander and Iwaniec to prove t-aspect subconvex bounds for degree two and three L-functions. We will also look at the subconvexity problem in the twist aspect, and in particular prove the Burgess bound for degree two L-functions. (A level lowering technique plays a crucial role in this delta symbol approach to subconvexity.) Next we will introduce the Bessel delta method which is tailor-made to tackle problems related to GL(2) Fourier coefficients. As an application we will study exponential sums involving such Fourier coefficients and prove a

sub-Weyl bound in this context. Finally we will consider delta methods which are derived from Petersson trace formula or Kuznetsov trace formula, and see some applications to spectral/weight aspect subconvexity problems.

- Simon Myerson (University of Warwick) Repulsion: a how-to guide (series)
  Abstract: Consider the integral zeroes of one or more, not necessarily diagonal, integral polynomials in many variables with the same degree. The basic principles for applying the circle method here were laid out by Birch. One way to improve on his work is repulsion: showing that the exponential sum over the polynomials can be large only on small, well separated regions. Unusually for improvements on Birch's work this idea has been successfully applied to systems which are not particularly singular and which contain many polynomials. To begin I will ask: what about Birch's work suggests that repulsion could be an improvement? I will then discuss the quadratic and higher degree case in detail, and an application to systems of forms with real coefficients.

- Lillian Pierce (Duke University) Applications of the circle method in harmonic analysis (series)
  Abstract: Around 1990, Bourgain realized that the circle method was an effective tool to understand the behavior of certain discrete operators in harmonic analysis, with interesting applications in ergodic theory. This opened the door to studying a wide range of discrete analogues of operators in harmonic analysis, by combining ideas of a dissection into major/minor arcs with the existing understanding of the associated real-variable operators. In this lecture series, we will introduce several exemplar problems of this type, provide some of the relevant analytic background, and demonstrate how ideas from the circle method play a role.

- Damaris Schindler (University of Göttingen) Beyond the circle method
  Abstract: The circle method is useful in counting integer solutions to systems of Diophantine equations. From a more geometric point of view the circle method provides us a valuable tool for counting rational points of bounded height on certain varieties. But what happens in situations where we don't have enough variables to apply the circle method? Or where we are not counting points on varieties but points close to manifolds? What if the height function on a projective variety does not directly lead us to a counting question in boxes, as is the situation where the circle method is best applied? These and related questions are going to be the topic of this talk. We are going to focus on situations where analytic tools play a key role in finding answers.

- Pankaj Vishe (Durham University) On the Hasse principle for complete intersections
  Abstract: Let X be a smooth projective complete intersection variety in $\mathbb{P}^{n-1}$ defined by a system of two cubic polynomials. We prove that X

satisfies the Hasse principle as long as $n \geq 40$. The key ingredient here is the development of a Kloosterman refinement for complete intersections over $\mathbb{Q}$. This is a joint work with Matthew Northey.

4 Number Theory and Harmonic Analysis Seminars

The two seminars ran over Zoom every one of the 12 Mondays of the trimester outside the duration of the summer schools (with one talk each of those Mondays, except for two last-minute cancellations in the analysis seminar). The talks were very popular, each attended by 50-80 people.

After most talks a Zoom breakout room was opened for the speaker and the participants who wanted to ask more specific questions/discuss more informally. The idea was to replicate the situation where people would approach the speaker at the board after the talk. These breakout sessions were also successful, with several interesting mathematical discussions and social catching-up.

Number Theory speakers: Regis de la Breteche (Paris), Paul Nelson (Zurich), Yiannis Sakellaridis (Johns Hopkins University), Claudia Alfes (Bielefeld), Maryna Viazovska (EPFL), James Maynard (Oxford), Matthew Young (Texas A&M), Sarah Peluse (Princeton/IAS), Alexandra Florea (Columbia), Ian Petrow (London), Rachel Greenfeld (UCLA), Will Sawin (Columbia).

Harmonic Analysis speakers: Ciprian Demeter (Indiana), Ruixiang Zhang (IAS), Mariusz Mirek (Rutgers), Jonathan Hickman (Edinburgh), Michael Christ (UC Berkeley), Po Lam Yung (ANU Mathematical Sciences, Betsy Stovall (UW Madison), Jim Wright (Edinburgh), Shaoming Guo (UW Madison).

5 Tao Lectures

Terence Tao was in residence in Bonn for part of the trimester, and gave a 3-part lecture series.

- Lecture 1: Pseudorandomness of the Liouville function
  This occurred on Opening Day; see abstract above.

- Lecture 2: Singmaster's conjecture in the interior of Pascal's triangle
  Abstract: An old conjecture of Singmaster asserts that every integer greater than 1 occurs only a bounded number of times in Pascal's triangle. In this talk we survey some results on this conjecture, and present a recent result in joint work with Kaisa Matomaki, Maksym Radziwill, Xuancheng Shao, Joni Teravainen that establishes the conjecture in the interior region of the triangle. Our proof methods combine an "Archimedean" argument due to Kane (and reminiscent of the Bombieri-Pila determinant method) with a "non-Archimedean argument" based on Vinogradov's exponential sum estimates over primes.

- Lecture 3: The circle method from the perspective of higher order Fourier analysis
  Abstract: Higher order Fourier analysis is a collection of results and methods that can be used to control multilinear averages (such as counts for the number of four-term progressions in a set) that are out of reach of conventional linear Fourier analysis methods (i.e., out of reach of the circle method). One notable feature of this theory is that the role of linear phase functions is replaced by the notion of a nilsequence. On the other hand, key identities from linear Fourier analysis, such as the Plancherel identity or the Fourier inversion formula, are notably absent in the higher order theory. In this survey talk we give an introduction to the higher order Fourier theory by revisiting the linear circle method from a higher order perspective, in particular downplaying as much as possible the role of Fourier identities.

6 On-site experience

The institute's environment allowed for a very vibrant mathematical experience. There was daily discussion and collaboration by several small groups of people on the institute grounds (garden blackboard, garden tables, balcony, common rooms blackboards). Moreover, the social aspect was particularly strong, and further enhanced mathematical communication. After a tumultuous year due to the pandemic, the institute offered a very much needed opportunity to a balanced life.

A special experience was an in-person talk by Sarah Peluse (one of the on-site speakers of the harmonic analysis seminars), which was live-streamed thanks to the institute's equipment.

7 Online experience

- Throughout the program, 5 **office hours** were held over Zoom, each hosted by an organiser and an external co-host (an early-career mathematician). During each office hour, two prominent mathematicians would answer questions from the audience, related to academic life overall (their academic experience and choices, successes and struggles, work-life balance, potential arising problems in academia, writing and reviewing papers, new journals).

  The aim of this initiative was to allow young researchers to get practical advice from senior mathematicians - advice that would be difficult to seek during a program mostly participated virtually. And indeed, the conversation was flowing during these hours, on each occasion ending due to lack of time, rather than of questions. The attendance was exceptionally high (even more so than for the seminars), with 70-120 participants each time.

  The details of the office hours were as follows:

  - Jul 2: Kannan Soundararajan (Stanford), Emmanuel Kowalski (ETH)
    Host: Valentin Blomer
    Co-host: Lola Thompson (Utrecht)
  - Jul 9: Malabika Pramanik (UBC), Andreas Seeger (Madison)
    Host: Philip Gressman
    Co-Host: Taryn Flock (Macalester)
  - Jul 23: Terence Tao (UCLA), Christoph Thiele (Bonn)
    Host: Lillian Pierce
    Co-host: Dóminique Kemp (Indiana)
  - Jul 30: Matthew Young (Texas A&M), Kaisa Matomaki (Turku)
    Host: Farrell Brumley
    Co-host: Jasmin Matz (Copenhagen)
  - Aug 13: Tony Carbery (Edinburgh), Betsy Stovall (Madison)
    Host: Marina Iliopoulou
    Co-host: Dominique Maldague (MIT)

- Discord: 90 connecting participants. Individual salons were made for each series of events, with announcements and Zoom links for all talks. Participants were encouranged to discuss the talks online in dedicated textual salons, and indeed many talks were followed by an active exchange on Discord. Moreover, we were able to answer practical questions by participants (often regarding Covid restrictions and protocols for onsite visitors), in a way which avoided mass emails, but which could collectively benefit the online community.

- Short video presentations: Surrounding live streamed talks, part of the trimester program consisted of prerecorded talks, where selected participants presented their research work in a condensed form. We are thankful to Ayla Gafni, Kevin Hughes, and Zane Li who each contributed a 10-12 minute talk for this event. Their videos are collected at the following webpage

  `https://www.him.uni-bonn.de/index.php?id=4980`

  We had a live question and answer discussion (via Zoom) on Friday, August 13, with each speaker that was attended by about 40 people.

8 Publications and ongoing resources

- For publications communicated to us by participants, please see the link here: Publications List

- The main webpage continues to provide links to slides and also videos of many of the lectures that occurred during the program: Main Webpage

Thank you for the support of this energetic dual-trimester program, and for the flexible administration of a hybrid online/in-person trimester.