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Report on the Trimester Program

Types, Sets and Constructions

May - August 2018
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Organizers: Douglas S. Bridges (Canterbury), Michael Rathjen (Leeds), Peter Schuster (Verona), Helmut Schwichtenberg (München).

Topics

The trimester brought together many of the leading people in type theory, proof theory, constructive set theory and constructive mathematics, and it has been a unique opportunity to discuss current topics at the research frontier.

The main subjects were:

1. Homotopy type theory and constructive models of univalence

2. Constructive mathematics and set theory

3. Extraction of computational information from proofs (aka proof mining)

4. Explicit mathematics and operational set theory

5. Structural proof theory

6. Ordinal analysis, ordinal representations and reverse mathematics

Goals

The aim of the Hausdorff Trimester was to create a forum for research on the many exciting recent developments, which are of central importance to modern foundations of mathematics, especially its constructive aspects. Among these are many developments in proof theory (e.g. proof assistants, proof mining, ordinal analysis, geometric logic), constructive analysis, algebra and economics, as well as the univalent foundations project initiated by the late V. Voevodsky that provides a new and rather unexpected semantics for type theories using notions from geometry and topology in order to model identity types.

A further impetus for having this trimester in 2018 was that it is 50 years after the edition of Bishop's seminal work *Foundations of Constructive Analysis*, with which he revolutionized constructive analysis and the foundations of mathematics.

Organization

During the Trimester we had a Summer School and four workshops. In addition there was a regular series of seminars, organized by Douglas Bridges.

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Types, Sets and Constructions. Summer School (Organizers: Douglas S. Bridges, Michael Rathjen, Peter Schuster, Helmut Schwichtenberg), May 3-9, 2018.

The following tutorials were given:

Peter Aczel (Manchester): Constructive set theory; Robert Constable (Cornell University): Proof assistants and formalization; Thierry Coquand (Göteborg): Constructive algebra; Peter Dybjer (Göteborg): Intuitionistic Type Theory; Martín Escardó (Birmingham): Constructive mathematics in univalent type theory; Matthew Hendtlass (Canterbury): Constructive analysis; Simon Huber (Göteborg): Homotopy type theory; Ulrich Kohlenbach (Darmstadt): Extraction of information from proofs; Isabel Oitavem (Universidade Nova de Lisboa): Recursion and Complexity; Andreas Weiermann (Gent): Higher proof theory and combinatorics.

Types, Homotopy Type theory, and Verification. Workshop (Organizers: Steve Awodey, Thierry Coquand, Maria Emilia Maietti), June 4-8, 2018.

Proofs and Computation. Workshop (Organizers: Rosalie Iemhoff, Gerhard Jäger, Andreas Weiermann), July 2-6, 2018.

Proof, Computation, Complexity. Workshop (Organizers: Reinhard Kahle, Ralph Matthes), July 19-20, 2018.

Constructive Mathematics. Workshop (Organizers: Douglas Bridges, Hajime Ishihara, Helmut Schwichtenberg), August 6-10, 2018.

Scientific outcomes

Owing to the enormous breadth of the research conducted during the Hausdorff trimester we can only provide a few snapshots, more or less haphazardly:

Several people worked on cubical models of homotopy type theory (e.g. Steve Awodey, Thierry Coquand, Anders Mörtberg Nicola Gambino, Pino Rosolini, Jonas Frey, Milly Maietti). Steve Awodey with others constructed a uniform, algebraic fibrant replacement operation on the category of cubical sets, making use of the corresponding operation for trivial Fibrations given by the partiality classifier.

Peter Dybjer collaborated with Simon Huber and Christian Sattler on syntactic models of intuitionistic type theory. They succeeded in defining several models, both intrinsically and extrinsically typed ones, and proved their isomorphisms.

Maria Emilia Maietti worked on developing a predicative generalization of the notion of elementary topos and of quasi-topos by building examples based on Martin-Löf's type theory, on Homotopy Type Theory and Voevodsky Univalent Foundations and on realizability within Hyland's Effective Topos. With Pino Rosolini and Giovanni Sambin she worked on a characterization of the models of the Minimalist Foundation which satisfy the Rule of Unique Choice.

Andrey Bauer worked with Peter Lumsdaine on general type theories. This is still ongoing work, but one particular piece of work that benefited from that is the Andromeda proof assistant (http://www.andromeda-prover.org). The work made it possible to extend the proof assistant so that it supports user-definable type theories.

Sara Negri during her stay at the Hausdorff institute concentrated on neighbourhood generalizations of possible worlds semantics for the proof-theoretical analysis of non-normal modal logics. This research has lead to two publications that were completed during her stay (see list at the end). Collaborative efforts on type theory and geometric logic that emerged from the Hausdorff trimester led to intensive and ongoing collaborations between Coquand, Negri, Rathjen, Schuster.

Andreas Weiermann, Tosi Arai, and D. Fernandez Duque worked on generalized Goodstein sequences (research published in the AMS Proceedings). This research initiated a new interpretation of ordinal notations in terms of canonical notations for numbers and this new theory connects naturally to the area of integer complexity. An innovative aspect of these investigations is that the Goodstein sequences will terminate to a large extent independently of the choice of normal forms which are used for representing numbers. Moreover the new insights on normal forms for integers form a key ingredient for proving far reaching comparisons between the slow and fast growing hierarchies.

Michael Rathjen and Andreas Weiermann worked on finitary treatments of infinitary proof theory, using work of Rathjen's former PhD student Michael Toppel. The work is important for proving extreme speed up theorems à la Friedman. Anton Freund, Weiermann and Rathjen worked on a general theory of Kruskal-like independence results. With Martin Krombholz, Rathjen worked on the proof-theoretic strength of the Robertson and Seymour graph minor theorem. Rathjen and Alec Thomson also gave a characterization of the existence of ω-models of Π_1^1-comprehension in terms of dilators, using Schütte-style search trees and an ordinal analysis based on Buchholz' Ω_n-rules. With Wilfried Sieg he wrote on a long paper on proof theory and with Peter Aczel he worked on models of constructive set theory in type theory. The latter work is concerned with providing realizability interpretations of the main formal system, CZF, for Constructive Set Theory. With Robert Lubarsky, Rathjen also worked on realizability for set theory, and with Reinhard Kahle on a book about Kurt Schütte's legacy in proof theory, involving many contributors staying at HIM.

Gerhard Jäger's research during his stay at HIM was concentrated on fixed point axioms and related principles in Kripke-Platek environments. In addition, he did some work on constructive principles in explicit mathematics. He obtained some new results about the relationship between several of the principles mentioned above. Also, he started some collaboration about constructive set theory in the framework of explicit mathematics.

Wilfried Sieg, after discussion with Coquand and Palmgren, was able to improve significantly a paper he had worked on with Patrick Walsh. It is called "Natural formalization: Proving the Cantor-Bernstein Theorem in ZF". An off-shoot of that paper, "The Cantor-Bernstein theorem: how many proofs?" was published in the Philosophical Transactions of the Royal Society.

Peter Schuster, Daniel Wessel and Ihsen Yengui worked on extracting computational content from classical proofs with Zorn's Lemma resulting in their paper "Dynamic evaluation of integrity and the computational content of Krull's lemma". This is very promising and

has led to very general results recently. In more detail, the paper addresses the following question: A multiplicative subset of a commutative ring contains the zero element precisely if the set in question meets every prime ideal. While this form of Krull's Lemma takes recourse to transfinite reasoning, it has recently allowed for a crucial reduction to the integral case in Kemper and the third author's novel characterization of the valuative dimension. They present a dynamical solution by way of which that characterization becomes fully constructive, and illustrate the method with concrete examples. They further give a combinatorial explanation by relating the Zariski lattice to a certain inductively generated class of finite rooted binary trees. In particular, the authors make explicit the computational content of Krull's Lemma.

Hajime Ishihara worked on implicit complexity theory, constructive reverse mathematics and constructive integration theory. With Josef Berger and Takako Nemoto, he worked on a form of Baire (category) theorem from constructive reverse mathematics point of view. They have found that the form of Baire theorem is closely related to a boundedness principle (BD-N), which is provable in classical, constructive recursive and intuitionistic mathematics. Although they could not prove that BD-N implies the form of Baire theorem, they are going to continue to work on this problem. With Samuele Maschio, they have worked on a constructive theory of integration.

Douglas Bridges and Matthew Hendtlass worked on constructive mathematical economics and wrote a paper. Bridges' half covers: the constructive definition and basic properties of preference relations; definitions of, and theorems about, local nonsatiation and continuity properties; a constructive proof of the Arrow-Hahn theorem on the existence of utility functions representing preference relations on locally compact convex metric spaces; and the continuity-in-parameters of the Arrow-Hahn utility function.

Bridges also worked, with R. Alps, on Constructive Morse set theory (CMST), a project to develop a constructive counterpart to the highly formalised set theory presented in A.P. Morse: A Theory of Sets, Academic Press, 1965. Being presented in a kind of pseudocode, CMST should facilitate both proof verification and program extraction from proofs. (Alps and Neveln, in the USA, are concurrently developing a proof verifier for CMST.)

Helmut Schwichtenberg worked on computational aspects of Bishop's constructive mathematics among other things, resulting in a paper that discusses a theory of computable functionals (closely related to HA^ω and Martin-Löf type theory) which puts the computational content of constructive proofs into the foreground. The main result is a soundness theorem, which states that the term t extracted from a constructive proof of a formula A realizes the proven formula. This term t is in an extension of Gödel's system T, and "t realizes A" is a formula of the theory, which is formally proved. Since t can be viewed as a program solving the "problem" (Kolmogorov) posed by the "specification" A, the soundness theorem provides a formal verification of the correctness of this program.

Schwichtenberg with Franziskus Wiesnet also worked on logic for exact real arithmetic, continuing earlier work of his with U. Berger, K. Miyamoto and H. Tsuiki. It is shown how a division algorithm for real numbers given as either a stream of signed digits or via Gray code can be extracted from an appropriate formal proof. The property of being a real number represented in either of these forms is formulated by means of coinductively defined predicates, and formal proofs involve coinduction. The proof assistant Minlog is used to

generate the formal proofs and extract their computational content, both as Scheme and Haskell programs.

The main topic of Iosif Petrakis' work at HIM was Bishop's constructive theory of sets and functions (BSFT), and especially the theory of families of sets and families of subsets within it. BSFT is the underlying set theory of Bishop-style constructive mathematics, one of the major topics of this trimester, that was only sketched by Bishop and hardly developed within Bishop-style constructive mathematics itself. He also had the opportunity to connect previous work on constructive topology to BSFT, by generalizing the notion of Bishop topology in order to be able to define a Bishop topology on families of subsets, and showing a Stone-Cech theorem for Bishop spaces.

A small sample of publications arising from the Trimester

Proof, Computation, Complexity

Bahareh Afshari, Gerhard Jäger, Graham E. Leigh, *An Infinitary Treatment of Full Mu-Calculus*. WoLLIC 2019: 17-34

T. Arai, D. Fernandez-Duque, S. Wainer, A. Weiermann, *Predicatively unprovable termination of the Ackermannian Goodstein process*, arXiv:1906.00020, 2019.

Marc Bezem and Thierry Coquand, *Skolem's Theorem in Coherent Logic*, Fundam. Inform.

S. Centrone, S. Negri, D. Sarikaya, P. Schuster (eds.), *Mathesis Universalis, Computability and Proof*, Springer 2019.

Tiziano Dalmonte, Nicola Olivetti and Sara Negri. *Non-normal modal logics: bi-neighbourhood semantics and its labelled calculi.* Advances in Modal Logic , 2018. Bezhanishvili, G., D'Agostino, G., Metcalfe, G. & Studer, T. (eds.). College publications, 20 p. (Advances in Modal Logic; vol. 12).

M. Krombholz, M. Rathjen, *Upper bounds on the graph minor theorem*, to appear in: *Well-Quasi Orders in Computation, Logic, Language and Reasoning*, Springer Verlag, Trends in Logic 51, edited by P. Schuster, M. Seisenberger, A. Weiermann

Sara Negri and Eugenio Orlandelli. *Proof theory for quantified monotone modal logics.* Logic Journal of the IGPL, Logic journal of the IGPL. 27, 4, p. 478-506.

Satoru Niki, Peter Schuster, *On Scott's Semantics for Many-Valued Logic* , Submitted.

T. Powell, P. Schuster, F. Wiesnet, *An algorithmic approach to the existence of ideal objects in commutative algebra*, Proceedings 26th Workshop on Logic, Language, Information and Computation (WoLLIC 2019).

M. Rathjen, W. Sieg, *Proof Theory*, Stanford Encyclopedia of Philosophy.

M. Rathjen, I.A. Thomson, *Well-ordering principles, ω-models and Π_1^1-comprehension*, to appear in: *The legacy of Kurt Schütte*, edited by K. Kahle et al.

M. Rathjen, M. Toppel, *On relating theories: Proof-theoretical reduction*, in: Mathesis Universalis, Computability and Proof, (eds. S. Centrone, S. Negri, D. Sarikaya, P. Schuster), Springer 2019, 311-332.

M. Rathjen, S. Tupailo, *On the Strength of the Uniform Fixed Point Principle in Intuitionistic Explicit Mathematics*, to appear in: *The legacy of Kurt Schütte*, edited by K. Kahle et al.

H. Schwichtenberg, *Program extraction from proofs: the fan theorem for uniformly coconvex bars*, in: Mathesis Universalis, Computability and Proof, (eds. S. Centrone, S. Negri, D. Sarikaya, P. Schuster), Springer 2019.

H. Schwichtenberg, F. Wiesnet, *Logic for exact real arithmetic*, Submitted to CCC19, http://arxiv.org/abs/1904.12763, 2019.

W. Sieg, *The Cantor-Bernstein theorem: how many proofs?*, Philosophical Transactions of the Royal Society A, volume 377, Issue 2140.

W. Sieg, P. Walsh. *Natural formalization: Proving the Cantor-Bernstein Theorem in ZF*, The Review of Symbolic Logic, 2019.

Type Theory and Constructive Mathematics

D.S. Bridges, *Intuitionistic sequential compactness?*, Indag. Math. 29(2018), 1477–1496.

D.S. Bridges, M. Hendtlass, *Constructive mathematical economics*, to appear in Handbook of Constructive Mathematics (Editors: D. Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg).

Thierry Coquand, *A survey of constructive presheaf models of univalence*, SIGLOG News.

Coquand, Thierry and Lombardi, Henri and Neuwirth, Stefan, *Lattice-ordered groups generated by an ordered group and regular systems of ideals*, Rocky Mountain J. Math.

K. Schlagbauer, P. Schuster, D. Wessel, *Der Satz von Hahn–Banach per Disjunktionselimination*, Confluentes Math., 2019.

Giulio Fellin, Peter Schuster, Daniel Wessel, *The Jacobson Radical of a Propositional Theory*. Submitted

Maroua Gamanda, Henri Lombardi, Stefan Neuwirth and Ihsen Yengui. *The syzygy theorem for Bézout rings*. Mathematics of Computation. DOI: https://doi.org/10.1090/mcom/3466.

P. Schuster, D. Wessel, I. Yengui, *Dynamic evaluation of integrity and the computational content of Krull's lemma*, 2019, Submitted.

H. Schwichtenberg, *Computational aspects of Bishop's constructive mathematics*, to appear in: Handbook of Constructive Mathematics (Editors: D. Bridges, H. Ishihara, M. Rathjen, H. Schwichtenberg).

Ihsen Yengui. *Computational Commutative Algebra and Algebraic Geometry*: Course and exercises with detailed solutions (year 2019). Kindle Direct Publishing. ISBN-10: 1096374447 ISBN-13: 978-1096374442

D. Wessel, *Point-free spectra of linear spreads*, in: Mathesis Universalis, Computability and Proof, (eds. S. Centrone, S. Negri, D. Sarikaya, P. Schuster), Springer 2019.