

Report on the Trimester Program

# Number Theory, Algebraic Geometry and Physics

January - April 2018

**Organizers:** Spencer Bloch (Chicago), Herbert Gangl (Durham), Vasily Golyshhev (Moscow), Fernando Rodriguez Villegas (Trieste), Don Zagier (Bonn)

## Topics

The word “period” is used to designate any number represented by the integral of an algebraic differential form over a cycle in an algebraic variety over  $\mathbb{Q}$  (or  $\overline{\mathbb{Q}}$ ). These include many numbers of interest in number theory and mathematical physics (multiple zeta values, Mahler measures, superstring amplitudes, ...), and also have deep connections with special values of motivic L-functions.

## Goals

The theory of mixed motives has come to play a central role in arithmetic algebraic geometry. The goal of the conference was to better understand how this abstract theory relates to questions in arithmetic, differential equations, and physics.

Some of the specific subjects that we wanted to study include the relation of periods of mixed Tate motives to MZV’s (Deligne, Goncharov, Brown); the conjectures of Beilinson, Deligne, Scholl on motivic L-functions; Picard-Fuchs differential equations and mixed motives (especially hypergeometric ones); mirror symmetry; mixed motives and superstring amplitudes. Many exciting new developments in the field were reported at or developed during the program and thereafter. A few prominent ones are listed here.

Ayoub presented a proof of the Conservation Conjecture for Chow motives, while Rudenko reported on his work with Goncharov proving the weight 4 case of Zagier’s Polylogarithm Conjecture, which served as inspiration for subsequent work by three other participants (Charlton, Gangl, Radchenko) on an explicit quadruple ratio and substantial progress on the weight 5 case.

Elliptic motives (Cao) and elliptic iterated integrals (Schneps, Matthes) as well as elliptic polylogarithms (Vanhove, Bloch, Broadhurst, Bogner, Weinzierl, Broedel, Duhr) played an important role for both the pure mathematics and physics activities. Furthermore, the connection of amplitudes and Feynman integrals to mirror symmetry of Calabi-Yau manifold (Vanhove, Kerr et al.) as well as to both open and closed string theory (Stieberger, Schlotterer, Weinzierl, Zerbini) were exploited.

A substantial part of the programme centered around the recent proof by Golyshev and Zagier of the Gamma conjecture for Fano 3-folds with Picard rank 1, leading to a new notion of “motivic Gamma functions” studied by Bloch and Vlasenko (and reported on in the afternoon seminar) as well as an unexpected relationship with multiple zeta values (see below).

Many different versions of motives were highlighted like relative 1-motives (André), Nori motives (Huber-Klawitter) or exponential motives (Fresan), and Galois groups of both “cosmic” nature (Dixon) and of mixed Hodge-Tate type (Zhu) made their appearance. Moreover, Brown and Dupont developed their single-valued integration method in connection with superstring amplitudes.

While the classical periods attached to a modular form were already well-known for 50 years (Eichler, Shimura), the far less familiar notion of quasi-periods of modular forms was studied independently and in completely different contexts by Brown and Hain and by Klemm, Scheidegger, Zagier, Golyshev and others (all but one of whom were heavily involved in the program). Other recent tools and constructions like modular arrangements (Levin) or modular inverters (Hain) gave further new insight into the area. Furthermore, two talks by Brown and Hain introduced multiple modular motives.

Early on in the activity Cohen, the original author of the powerful computer algebra package Pari/GP which is especially well tailored to questions in Number Theory, introduced a whole new package on modular forms. This should make explicit computations far more convenient for the “average user”.

Multiple zeta values in many of their avatars (classical (Hirose–Sato, Tanaka, Charlton),  $p$ -adic (Jarossay), single-valued (Stieberger), symmetric and  $q$ -deformed (Bachmann), elliptic (Matthes, Schneps, Zerbini), alternating (Hoffman), as coefficients in deformation quantization (Panzer), even in connection with Picard–Fuchs equations (Golyshev–Zagier)) were investigated thoroughly during the program. Enriquez and Furusho found a Betti counterpart of one of the families making up the famous double shuffle relations, while Scholl and Nekovar reported on a recently introduced (“plectic”) cohomology theory.

Special values of zeta functions (Lichtenbaum, de Jeu, Lemma, Flach, Zudilin, Harder, Maillot) and their connection to Mahler measures (Zudilin again) and mixed Tate motives (Dupont) played an important role, as did polylogarithms in infinitesimal (Unver) or graphical (Schnetz) form and even in connection with computational “Chabauty–Kim” (Dan-Cohen).

As a further bonus, a very recent proof of the triviality of  $K_8(\mathbf{Z})$ , a long-standing problem, was reported during the program (Elbaz-Vincent, presented in Manin’s seminar).

## Organization

1) **The first month** (Jan 3 - Feb 4) was dedicated to motives, periods and regulators, with a workshop as its “centerpiece” (Jan 15–Jan 19).

### **Workshop: Periods and Regulators**

Speakers were Yves André, Matt Kerr, Sinan Unver, Daniil Rudenko, Richard Hain, Joseph Ayoub, Marc Levine, Annette Huber-Klawitter, Jose Burgos,

Tomohide Terasoma, Hidekazu Furusho, Benjamin Enriquez, Michael Hoffman, Steven Charlton, Nobuo Sato, Anthony Scholl, Francesco Lemma, Stephen Lichtenbaum, Matthias Flach, Neil Dummigan, Jin Cao, Guangyu Zhu, Nils Matthes and Minoru Hirose.

2) **The second month** (Feb 4 - Mar 2) focused on the connection with amplitudes in Quantum Field Theory, culminating in a second workshop (Feb 26–Mar 2).

**Workshop: Amplitudes and Periods**

Speakers were David Broadhurst, Erik Panzer, Christian Bogner, Vyacheslav P. Spiridonov, Dirk Kreimer, Ralph Kaufmann, Oliver Schnetz, Johannes Bluemlein, Stephan Stieberger, Claude Duhr, Helene Esnault, Francis Brown, Oliver Schlotterer, Federico Zerbini, Pierre Vanhove, Stefan Weinzierl, Johannes Brödel, Axel Kleinschmidt, Lance Dixon.

3) The main emphasis of **the third and fourth month** were on Picard-Fuchs Equations and Geometry and on Hypergeometric Motives, respectively, with a workshop mid-way which highlighted the close relationship of these topics with each other (Mar 26–Mar 30).

**Workshop: Picard-Fuchs Equations and Hypergeometric Motives**

Speakers were Frits Beukers, Alexander Varchenko, Bartosz Naskrecki, Roberto Villaflor Loyola, Mark Watkins, Madhav Nori, Wadim Zudilin, Masha Vlasenko, Jan Stienstra, R. Paul Horja, Duco van Straten, Kiran S. Kedlaya, Danylo Radchenko, Damian Rössler, Robert Kucharczyk, John Voight.

4) **Seminar talks.** Apart from these concentrated periods there was an officially announced **Trimester Seminar**: Speakers included Andrey Levin, Don Zagier, Spencer Bloch, Vasily Golyshev, Hiroshi Iritani, Matthew Kerr, Mikhail Bondarko, Rob de Jeu, Fernando Rodriguez Villegas, Henri Cohen, Francis Brown, I.D. Shkredov, Clement Dupont, Werner Nahm, Marianne Leitner, Jan Stienstra, Jean-Baptiste Teyssier, Haoyu Hu, Duco van Straten, James Lewis, Michael Groechenig, Alexander Varchenko, Gonzalo Tornaria, David Roberts, Shehryar Sikander, Wadim Zudilin, Ishai Dan-Cohen, David Jarossay, Vadim Vologodsky, Günter Harder, Marius van der Put, Di Yang.

Talks were coordinated with the regular seminars at the MPI with similar topics: the **Number Theory Lunch Seminar** as well as Yuri Manin's **Seminar on Algebra, Geometry and Physics** which resulted in a good exchange of speakers and topics: Speakers included Kumar Murty, Dinakar Ramakrishnan, Anton Mellit, Vincent Maillot, Oliver Lorscheid, Hohto Bekki, Javier Fresan, Francesco Baldassarri, Philippe Elbaz-Vincent.

5) **“What is” talks:** In lieu of a school there were introductory lectures to the activities of the trimester program, mainly directed at PhD students and young postdocs. Moreover, there were plenty of “What is” talks whose aim was to give a gentle introduction to rather specific notions that are central to the respective theories. These latter talks were given by Yves André, Richard Hain,

Leila Schneps, Dinakar Ramakrishnan, Masha Vlasenko, Matthias Flach, Javier Fresan, Oliver Schnetz, Piotr Achinger and Madhav Nori.

6) **MZV days.** We organized yet another 2-day workshop on Multiple Zeta Values, called the “MZV days”, the speakers of which were almost all young postdoctoral students: Henrik Bachmann, Tatsushi Tanaka, Nils Matthes, Koji Tasaka, Steven Charlton, Minoru Hirose, Nobuo Sato,

During the activity two further conferences took place whose topics were thematically very closely related: on Feb 19-23 at the Bethe Forum Scattering on “Amplitudes in Gauge Theory, Gravity and Beyond”, and Mar 12-14 the “Conference on Arithmetic and Automorphic Forms on the occasion of Günter Harder’s 80th birthday”.

7) Days with no official talks in the afternoon seminar were dedicated to **joint work**, which focused primarily on geometric variations of mixed Hodge structures and their relationship to motivic gamma functions, the Beilinson conjectures, and quantum differential equations of local Calabi–Yau manifolds.

## Results

For us, the impact of the conference is measured by the amount of subsequent work in the subject by conference participants. We are delighted both by the volume of work and by the diversity of areas of math and physics influenced by these ideas.

A partial list of the many publications and preprints arising from or developed during the activity can be found by clicking on the link here in the pdf: <https://tinyurl.com/himperiodspubl>.