Report of Group C

Scale separation of gradient flows in continuous and discrete setting

Junior Hausdorff Trimester Program

Optimal Transportation

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The topics of the projects in our group on "scale separation of gradient flows in continuous and discrete setting" consisted in the investigation of gradient flows, their derivation and possible limits. We investigated the qualitative behavior of gradient flows in terms of the underlying metric structure. In this spirit, we showed convergence of gradient flows in some system parameters like size or numbers of particles and its long-time behavior by quantified curvature bounds.

Projects and collaborations

- Manh Hong Duong, and Julian Tugaut [DT] studied the stationary solutions of the Vlasov-Fokker-Planck (VFP) system, which is a generalization of the McKean-Vlasov equation taking in addition a friction force into account. They prove, under suitable assumptions, that the VFP equation does not have a unique stationary solution and that there exists a phase transition. This is an important preliminary step to further study the long-time behavior of this system.

- Matthias Erbar, Max Fathi, Vaios Lachos and André Schlichting collaborated in a project [EFLS] during the trimester program on the gradient flow structure of McKean-Vlasov equations on discrete spaces. They showed that a family of non-linear mean-field equations, arising in statistical mechanics, can be viewed as a gradient flow of a natural free energy functional with respect to a certain metric structure. This is a generalization of former results by [3, 4] to nonlinear equations. In addition, they prove that this gradient flow structure arises as the limit of the gradient flow structures of a natural sequence of weakly interacting N-particle dynamics, as N goes to infinity.

- In a fellow-up work in progress, Matthias Erbar, Max Fathi, and André Schlichting investigate curvature properties of mean-field dynamics in the sense of [1]. Models in this category are mean-field Ising models like the Curie-Weiss magnet. These model show a phase transition, when the temperature crosses a critical value, which in statistical mechanics is characterized as the breakdown of convexity of the associated free energy. Preliminary results show, that the curvature changes it sign at exactly the same temperature.

- Max Fathi and Marielle Simon [FSi] present a new approach to the well-known convergence to the hydrodynamic limit for the symmetric simple exclusion process (SSEP). More precisely, they characterize any possible limit of its empirical density measures as solutions to the heat equation by passing to the limit in the gradient flow structure of the particle system.

- André Schlichting started and recently finished a project [Sch], in which he considers gradient structures for the Becker–Döring equation and its macroscopic limits. The result of Niethammer [5] is extended to proof the convergence not only for solutions of the Becker–Döring equation towards the Lifshitz–Slyozov–Wagner equation of coarsening, but also the convergence of the associated gradient structures. The gradient structure of the nonlocal coarsening equation is rigorously established. Further, it is proven that on the considered time scale the small cluster distribution of the Becker–Döring equation follows a quasistationary distribution dictated by the monomer concentration.

  The techniques of convergence of gradient flows in the projects [EFLS, FSi, Sch] is based on [6] and the involved authors took advantage in many discussions and exchange of ideas during the trimester program.

- Matthias Erbar and Nicolas Juillet [EJ] started a collaboration concerned with flows in the space of metric measure spaces during the trimester program. The project could be finalized during the follow-up workshop. They study a transformation of metric measure spaces introduced by Gigli and Mantegazza [2] consisting in replacing the original distance with the transport distance between heat kernel measures. For smooth manifolds this flow is tangent to the Ricci flow at time zero. They study the smoothing effect of this procedure in two important examples. In the case of specific Euclidean cones, they show that a singularity persists at the apex. They generalize the construction to the Heisenberg group as an example of a sub-Riemannian manifold, and show that the space is regularized instantaneously to a smooth Riemannian manifold.

- Further projects by Max Fathi [FSh, FSt] and Emmanuel Indrei [IM, Ind] took advantage of the excellent working conditions, scientific atmosphere and fruitful discussions over the cake and coffee break.

Activities organized by the group

The research group took advantage of the winter school organized at the very beginning of the trimester program. Lecture series by Marek Biskup (UCLA), Gero Friesecke (TU München), Jan Maas (IST Austria) and Mark Peletier (TU Eindhoven) gave an introduction to fields directly or indirectly related to the overall research scheme of our group.

Thanks to the excellent organization and help by the administration of the HIM, our group was able to organize two workshops. The first had as a topic *Analytic approaches to scaling limits for random system* and offered many especially young researchers the possibility to present their results. In addition, it brought together many collaborators of participants of the trimester program to foster collaborations. Among these were in particular Richard Kraaij (TU Delft), Tony Lelièvre (Ecole des Ponts ParisTech), Michiel Renger (Weierstrass Institute), Upanshu Sharma (TU Eindhoven), Gabriel Stoltz (Ecole des Ponts ParisTech) and Julian Tugaut (Université Jean Monnet).

The second workshop on *Gradient flows and entropy methods* consisted of a lecture series by Giuseppe Savaré (Università di Pavia) and talks given by Virginie Ehrlacher (CERMICS Ecole des Ponts et Chaussées), Julian Fischer (MPI for Mathematics in the Sciences), Stefano Lisini (Università di Pavia), Daniel Matthes (TU München), Christian Seis (Universität Bonn), Yan Shu (Université Paris Ouest Nanterre La Défense), Dejan Slepčev (Carnegie Mellon University), Martin Slowik (TU Berlin), and Julian Tugaut (CNRS).

Both workshops were successful in bringing together researchers and starting fruitful discussions and collaborations.

Publications and preprints

Legend: participants and guests of the JHTP.

[DT] M.H. Duong, J. Tugaut. Stationary solutions of the Vlasov-Fokker-Planck equation: existence, characterization and phase-transition. *Applied Mathematics Letters,* 52:38–45, feb 2016.

[EFLS] M. Erbar, M. Fathi, V. Laschos, and A. Schlichting. Gradient flow structure for McKean-Vlasov equations on discrete spaces. *To appear in Discrete and Continuous Dynamical Systems - Series A,* 36(12), dec 2016. *arXiv:1601.08098.*

[EJ] M. Erbar, and N. Juillet. Smoothing and non-smoothing via a flow tangent to the Ricci flow. *arXiv:1603.00280.*

[FSh] M. Fathi, and Y. Shu. Curvature and transport inequalities for Markov chains in discrete spaces. *To appear in Bernoulli. arXiv:1509.07160.*

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[FSt] M. Fathi, and G. Stoltz. Improving dynamical properties of stabilized discretizations of overdamped Langevin dynamics. *To appear in Numerische Mathematik. arXiv:1505.04905.*

[Ind] E. Indrei. A sharp lower bound on the polygonal isoperimetric deficit. *Proceedings of the American Mathematical Society*, 144:3115–3122, 2016

[IM] E. Indrei, and A. Minne. Nontransversal intersection of free and fixed boundaries for fully nonlinear elliptic operators in two dimensions. *Analysis & PDE*, 9(2):487–502, 2016.

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Report of group B

Optimal Transport and Stochastic Analysis

Junior Hausdorff Trimester Program on **Optimal transport**

January – April 2015

Goals and Group

The broad goal of our group was to push the development of (constrained) optimal transport problems with applications in stochastics with a special focus on martingale optimal transport and embedding problems. They have a natural interpretation in mathematical finance where these optimisation problems correspond to worst case scenarios. Using ideas of optimal transport in discrete time it is possible to derive general duality results which can be interpreted as superhedging results in mathematical finance. Moreover, establishing analogues of cyclical monotonicity it is often possible to explicitly construct and analyse primal optimisers which correspond to extremal models for the market in mathematical finance.

The challenge that we put ourselves for this trimester program was to extend (parts of) these results to continuous time. In parts this can be achieved by the transport approach to the Skorokhod embedding problem [BCH16b]. However, to obtain a powerful and general dual formulation as well as to provide geometrical characterizations of optimizers it is necessary to consider stochastic analysis in a pathwise manner. In particular, it is necessary to build on a powerful notion of a pathwise stochastic integral. We can subsume our goals under the following (general) problems:

Problem 1. Find a general method to calculate sharp robust bounds for option prices, incorporating market information at multiple intermediate time steps as well as historical data.

Problem 2. Develop tools that allow to analyze continuous time martingale optimal transport. In particular, develop a pathwise stochastic calculus appropriate for the challenges of martingale optimal transport.

Problem 3. Given some functional γ, find a general scheme to solve and explicitly describe (eg via PDEs) solutions to

$$\mathbb{E}[\gamma((B_s)_{s\leq\tau})] \to \max \qquad \text{subject to} \qquad B_\tau \sim \mu, \tau \in \mathfrak{C}, \tag{1}$$

where $\mathfrak{C}$ denotes a set of additional (eg distributional) contraints on τ.

We tackled these problems with a group of in total 9 people (2 of whom stayed for roughly a week, 5 for roughly a month, and 1 for two month and 1 for the whole period). Our group: Beatrice Acciaio, Mathias Beiglböck, Alex Cox, Martin Huesmann, Marcel Nutz, Harald Oberhauser, Nicolas Perkowski, David Prömel, and Pietro Siorpaes.

Results

Progress on Problem 1 (see also Progress on Problem 3):

[BCH+15]: Building on and extending the duality theory for Skorokhod embedding developed in [BCH16b] together with the game theoretic approach to finance by Vladimir Vovk (cf. lecture series) M.B., A.C., M.H., N.P. and D.P. were able to prove a general multi-marginal superhedging result in continuous time.

[AL15]: During the program B.A. and Martin Larsson (invited for the workshop) made significant progress on their study of semi-static completeness in a robust framework.

Progress on Problem 2:

[Vov16]: The game-theoretic approach of Vovk provides a set-up for financial mathematics that does not require any measure theoretic foundations. A natural question to ask is if there is any link between this approach and the more commonly used 'model free' approach, where the worst case over all possible probability measures is considered. We have discussed this question with V. Vovk, albeit without clear conclusions. However, these discussions motivated Vovk to keep thinking about the problem and after the trimester he was able to give a positive answer, at least under strong assumptions.

[ŁPP16]: N.P. and D.P. have discussed game-theoretic (and thus model free) constructions of the Itô integral. These constructions are based on an approach that was developed by V. Vovk (c.f. lecture series). Previously, they had obtained a construction in the special case of continuous integrators, and in subsequent work after our departure from HIM they were able to extend this to a variety of settings allowing also for jumps, and they even strengthened the result in the case of continuous integrators.

[FP16]: During the lecture course of Peter Friz, Peter Friz and D.P. initiated an extension of the continuity of the famous Lyons-Itô map to Besov-Nikolskii topologies.

Progress on Problem 3:

[BHS15]: M.B., M.H. and Florian Stebegg (visiting for the workshop) gave a new proof of Kellerer's celebrated theorem on the existence of Markov-martingales with a continuum of prescribed marginals by establishing that the set of all martingale measures with these marginals carries a natural compact Polish topology. A key step in the proof is the observation that the martingale coupling induced by the Root embedding has a Lipschitz-Markov kernel.

[BCH16a]: M.B., A.C., and M.H. managed to extend the framework of [BCH16b] to the case of multi-marginal Skorokhod embedding, ie to solve (1) with the constraint that to each candidate stopping time τ there are $n-1$ ordered stopping times $\tau_1 \leq \ldots \leq \tau_{n-1} \leq \tau$ each embedding an a priori fixed measure μ_i. By an application of the Dambis-Dubins-Schwarz theorem this result gives also the first construction and characterisation of solutions to the multi-marginal martingale optimal transport problem. Morevoer, by an application of [BHS15], this leads to the construction of optimal PCOC's.

[ACH16]: B.A., A.C., and M.H. developed an approach to insider information/trading based on Skorokhod embedding. Mathematically, this leads to constrained embedding problems, i.e. problems of the form (1). Under some conditions on the set $\mathfrak{C}$ they are able to show a general duality theory as well as the characterisation of optimisers similar in spirit to [BCH16b]. Using the approach of [BCH+15] this leads to superhedging results for the insider.

Activities

During the time at the HIM we organized

- the workshop *Optimal transport and stochastics* bringing together various researchers working on (constrained) optimal transport problems with applications in stochastic analysis, probability, statistics, and finance;
- a lecture series on *Rough Path* by Peter Friz;
- and a lecture series on *Game theoretic probability* by Vladimir Vovk; both lecture series were aiming at introducing powerful tools for pathwise stochastic calculus.

The HIM

The Hausdorff institute offered a nice and stimulating atmosphere and excellent working conditions. We are still benefitting from the stimulus of this program and looking forward to the next opportunity to spend some time at the HIM.

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[Vov16] Vladimir Vovk. Another example of duality between game-theoretic and measure-theoretic probability. *arXiv:1608.02706*, 2016.

JUNIOR HAUSDORFF TRIMESTER PROGRAM: OPTIMAL TRANSPORT (GROUP A)

JANUARY 5 - APRIL 24, 2015

Participants: Lashi Bandara (University of Gothenburg) 05.01.-24.04., Fabio Cavalletti (University of Pavia) 02.02.-28.03., Qintao Deng (Central China Normal University) 05.01.-24.04., Asma Hassannezhad (Max-Planck Institute Bonn) 01.02.-24.04., Heikki Jylhä (University of Jyväskylä) 05.01.-24.04., Martin Kell (Max-Planck-Institute Leipzig) 02.03.-24.04., Christian Ketterer (Universität Freiburg) 01.02.-24.04., Yu Kitabeppu (Kyoto University) 05.01.-31.03., Sajjad Lakzian (Universität Bonn) 05.01.-24.04., Andrea Mondino (ETH Zürich) 05.01.-14.02., Michael Munn (University of Missouri) 05.01.-24.04.

This is a report on the scientific activities of Group A (Geometry) at the Junior Hausdorff Trimester Program Optimal Transport 2015 in Bonn.

Group A studied global and local properties of non-smooth metric measure spaces. An important role plays the notion of synthetic lower Ricci curvature bounds that are defined via optimal transport techniques. This approach was initiated by celebrated works of Lott/Villani [LV09] and Sturm [Stu06a, Stu06b], and it is a non-smooth generalization of the classical notion of lower bounded Ricci tensor in smooth Riemannian geometry.

In subgroups the members started several collaborations on significant research projects, e.g. Levi-Gromov isoperimetric inequalities, existence of geometric flows for singular spaces, classification of low dimensional metric measure spaces. HIM offered possibilities to host several short time visitors who contributed with their expertise and creativity to these projects. Visitors were Fernando Galaz-Garcia (Karlsruher Institut fr Technologie), Alex Amenta (Universit Paris-Sud) and Shouhei Honda (Kyushu University).

In the course of the trimester program the group organized a workshop (New developments in Optimal Transport, Geometry and Analysis) consisting of 3 lectures courses (Nicola Gigli: Spaces with Ricci curvature bounded from below, Christina Sormani: A course on intrinsic flat convergence, Emanuel Milman: 1-D localisation) and 4 research talks (Shouhei Honda: Elliptic PDE on compact Ricci limit spaces and applications, Yashar Memarian: A Brunn-Minkowski type inequality on the sphere, Ionel Popescu: Free functional inequalities on the circle, Tapio Rajala: Tangents and dimensions of metric spaces). The workshop's aim was to present the state of the art in non-smooth geometric analysis and differential geometry for metric and metric measure spaces.

In the following we describe in more detail the scientific outcome of Group A during the Junior Trimester Program. Several collaborations have been initiated.

A. Levy-Gromov isoperimetric inequality. In [CMa] Fabio Cavalletti and Andrea Mondino develop a method for metric measure spaces to prove the celebrated Levy-Gromov isoperimetric inequality (and its generalization to arbitrary lower bounds established by E. Milman). They use the so-called 1-D localization (or needle decomposition) proposed in a recent work of B. Klartag in smooth Riemannian manifolds. This method has its roots in a work of Payne-Weinberger (1960) and was developed by Gromov-Milman and Kannan-Lovasz-Simonovits in papers of the 80-90'ies. The rough idea is to reduce the problem of establishing geometric inequalities to 1-dimensional problems. In his lecture during the workshop E. Milman gave a general introduction to 1-D localization and its applications. Cavalletti and Mondino also applied their ideas to prove sharp estimates for the p-Laplace operator and the Brunn-Minkowski inequality [CMb].

B. Rough metrics and regularity of the Gigli-Mantegazza flow. In [BLM, Ban] Lashi Bandara, Sajjad Lakzian and Michael Munn study a new geometric flow that was introduced by Gigli and Mantegazza in [GM14]. This flow is closely related to Hamilton's Ricci flow and has the special feature that it also exists for non-smooth initial geometries. Bandara, Lakzian and Munn study the Gigli-Mantegazza flow when the initial geometry is a topological manifold equipped with a Riemannian metric with measurable and bounded coefficients. This framework captures a wide class of non-smooth metric spaces and in particular the situation when the initial space has a cone-like singularity. They connect the regularity of the flow to the regularity of the initial heat kernel to study smoothening properties of the flow for explicit non-smooth inital data.

C. Variable lower Ricci curvature bounds. In [Ket] Christian Ketterer introduces a curvature-dimension condition for metric measure spaces with variable lower Ricci curvature bounds. It extends the aproach of Lott, Sturm and Villani for constant lower Ricci curvature bounds and previous work of Sturm in infinite dimensional context. A consequence is a sharp, generalized Bonnet-Myers theorem for metric measure spaces. Ketterer also proves stability and tensorization results, and compatibility with other notions of variable lower Ricci curvatures bounds.

D. Alexandrov spaces with lower Ricci curvature. In [DGGGM] Quintao Deng, Fernando Galaz-Garcia, Luis Guijarro and Michael Munn study the class of Alexandrov spaces that satisfy a positive or non-negative generalized lower Ricci curvature bound in the sense of Lott, Sturm and Villani. Alexandrov spaces are metric spaces that admit synthetic lower sectional curvature and therefore provide more regularity for the local structure. Deng, Galaz-Garcia, Guijarro and Munn were able to give a full topological classification for spaces with positive or non-negative Ricci curvature bounds in dimension 3.

E. Curvature-dimension condition in low dimensional context. In [KL] Yu Kitabeppu and Sajjad Lakzian study the class of metric measure spaces satisfying a Riemannian curvature-dimension condition $RCD^*(K,N)$ with low dimension bound N. If N is smaller than 2, they give a full classification, and in particular obtain that any such metric measure space arises as Gromov-Hausdorff limit of a sequence of Riemannian manifolds. This answers a common conjecture in low dimension. As by-product they deduce in any dimension a Bishop-Gromov-type inequality for the boundary measure of balls.

F. Geometric characterisation of L^∞-optimal transport. In [JR] Heikki Jylhä and Tapio Rajala study the L^∞-optimal transport distance in the context of general metric spaces. Though it was important in many areas of mathematics, the L^∞-transport distance is rather unexplored so far, and in contrast to its L^p-brothers little was known about its topology and its relation to weak convergence of measures. The main result of Jylhä and Rajala states that the L^∞-optimal transport distance between two probability measures can be estimated by their total cost with respect to a cost function that is a monotone function of the distance provided that the first measure has compact and connected support and the second measure is concentrated on the support of the first.

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