Report of

Daniel Greb, Anne-Sophie Kaloghiros, Vladimir Lazić, Chenyang Xu

Participants of the Junior Hausdorff Trimester Program

Algebraic Geometry
Group: "Birational and Hyperkähler Geometry"

Periods of stay:
Greb: 11.02.2014 – 15.04.2014
Kaloghiros: 17.02.2014 – 25.04.2014
Lazić: 06.01.2014 – 25.04.2014
Xu: 06.01.2014 – 06.03.2014

The "Birational and hyperkähler geometry" group had a fruitful and productive time during the Bonn Junior Trimester 2014. We collaborated on several of the projects outlined in our proposal for the Trimester. In addition, we organised several events that were very succesful; in our opinion, these raised the profile of the Trimester and of the Hausdorff Institute in the algebraic geometry community.

Projects

Our original goal was to make progress on the following five projects.

A. The Minimal Model Program. The main outstanding problem in the Minimal Model Program is the Abundance conjecture, which predicts that the canonical bundle of a minimal variety is semiample. A more general version of Abundance is to show the existence of good minimal models (minimal models on which the Abundance holds) for varieties with the pseudoeffective canonical bundle. Our goal in the Trimester was to make progress on the easier problem of existence of good models for varieties with non-negative Kodaira dimension.

B. Building blocks of varieties with trivial canonical class. Assuming the MMP for varieties of Kodaira dimension zero, the next natural step is to

analyse the structure of varieties X with canonical singularities and trivial canonical class. A natural question to ask is to what extent the Beauville-Bogomolov-Fujiki decomposition theorem generalises in this setting. Our aim was to identify potential building blocks of varieties with canonical singularities and trivial canonical class.

C. Cones of divisors on Calabi-Yau manifolds. The Cone conjecture of Morrison and Kawamata predicts that the geometry of a Calabi-Yau manifold is controlled by the action of automorphisms and birational automorphisms on the nef and movable cones. It implies that the number of minimal models of any variety with mild singularities is finite up to isomorphism, and it predicts existence of rational curves on Calabi-Yau threefolds, at least when the automorphism group is finite. One of our aims in this part of the project was to establish more cases of the Cone conjecture in dimension 3.

D. Hyperkähler manifolds. Two of the main open problems concerning the geometry of hyperkähler manifolds are the hyperkähler Strominger-Yau-Zaslow conjecture and the classification of base spaces for holomorphic Lagrangian fibrations. We planned to attack some special cases of the generalised hyperkähler SYZ conjecture, and the classification of base manifolds of Lagrangian fibrations from the MMP point of view.

E. Uniruled log Calabi-Yau varieties. If (Y, D) is a klt pair with $K_Y + D \sim_{\mathbb{Q}} 0$, then (Y, D) is a log Calabi-Yau. We planned to study log Calabi-Yau varieties (Y, D) that are compactifications of algebraic tori $T = Y \setminus D$. Given two such pairs (Y, D) and (Y', D') and a birational map $\varphi\colon Y \dashrightarrow Y'$ with $\varphi_* D = D'$, the goal was to investigate conditions under which φ can be decomposed into Sarkisov links which preserve the volume form.

Collaborations

We made progress on Projects A, C, D and E from our proposal. Due to time constraints, we have not been able to make advances on our Project B, but we are committed to working on it in the future.

On Project A, we have had a concentrated effort to, above all, understand the obstacles to different approaches to the Abundance conjecture in higher dimensions. To this end, in the period 10-21 February 2014 we organised a study group on the proof of the Abundance conjecture in dimension 3 by Miyaoka and Kawamata. The aim of the study group was to understand

how much of the proof can be generalised to higher dimensions, taking into account the advances in birational geometry in the 20 years since that proof was completed. The proof was presented in a series of lectures by Chenyang Xu and Vladimir Lazić. The participants also included several members of the mathematics department and of the Max Planck Institute.

On Project C, towards the end of the trimester Artie Prendergast-Smith and Vladimir Lazić started thinking about possible generalisations of the paper [LP12] into different contexts, namely when we consider a log Calabi-Yau pair and varieties of higher rank. Some initial steps were made, and may result in a collaboration in the future.

On Project D, we concentrated on proving that the base of a Lagrangian fibration of a hyperkähler 4-fold is the projective plane. We have made substantial effort towards this goal, and (partly in discussions with Sönke Rollenske and other participants of the program) we have developed several strategies which deepened our understanding of this difficult problem. All these strategies employ recent advances in the Minimal Model Program, and it is an exciting prospect that the MMP techniques can be used further in hyperkähler geometry, as implied by recent works [GLR13, Mat14]. This is a collaboration between Daniel Greb, Anne-Sophie Kaloghiros and Vladimir Lazić.

As an offshoot, this gave connections with our Project E. Indeed, the base of a Lagrangian fibration is, in a natural way, a Fano variety which is, at the same time, a log Calabi-Yau pair. One of our approaches to Project D involves using Sarkisov Program on the base, and the connection to Project E is immediate. We look forward to exploring this connection in the future work.

Resulting papers

Our activities during the Trimester resulted in the following papers.

Daniel Greb, *Complex-analytic quotients of algebraic G-varieties*, Math. Ann. 363 (2015), no. 1-2, 77-100.

Tobias Dorsch and Vladimir Lazić, *A note on the abundance conjecture*, Algebraic Geometry 2 (2015), no. 4, 476-488.

Alessio Corti and Anne-Sophie Kaloghiros, *The Sarkisov program for Mori fibred lc Calabi-Yau pairs*, arxiv:1504.00557.

Johannes Nicaise and Chenyang Xu, *Poles of maximal order of motivic zeta functions*, to appear in Duke Math. J., arXiv:1403.6792.

Activities organised by the group

We organised several highly successful activities during the Bonn Junior Trimester 2014.

In the period 6-22 January 2014 we organised, together with Daniel Huybrechts, the visit of Robert Lazarsfeld (Stony Brook University) as a Felix Klein lecturer. Even though this is a traditional yearly event at the Hausdorff Center for Mathematics, Daniel Huybrechts and we thought that the speaker fits perfectly within the scope of the Trimester. The topic was on the recent advances in the syzygy theory on projective varieties, and was an event which brought all the groups of the Trimester together.

The central event of our group at the Trimester was the workshop *Birational geometry and foliations*, in the period 24-28 February 2014. This was again a highly successful event, at which we had members of all groups at the Trimester as participants. The speakers included world's leading experts in birational and hyperkähler geometry, and the workshop was on par with any other similar event in the world. The speakers were: Arend Bayer, Frédéric Campana, Paolo Cascini, Alessio Corti, Jun-Muk Hwang, Stefan Kebekus, Sándor Kovács, Brian Lehmann, Christian Lehn, Christian Liedtke, James M^{c}Kernan, Mircea Mustaţă, Gianluca Pacienza, Thomas Peternell, Taro Sano, Karl Schwede and Mingmin Shen.

Finally, we had three short-term visitors: Thomas Eckl (25-28 March 2014), Junyan Cao (30 March-4 April 2014) and John-Christian Ottem (6-12 April 2014). With Eckl we discussed recent analytic techniques of Siu related to the Abundance conjecture; with Cao we discussed analytic techniques related to extension of sections from subvarieties and the Ohsawa-Takegoshi theorem; with Ottem we discussed recent developments on the behaviour of subvarieties of projective varieties with ample normal bundles. The visitors gave seminars in the realm of the Trimester Seminar session, which we also shared with the usual Seminar Algebraic Geometry of the University of Bonn: Eckl spoke on 27 March, Cao on 3 April, and Ottem on 10 April 2014.

References

[GLR13] D. Greb, C. Lehn, and S. Rollenske, *Lagrangian fibrations on hyperkähler manifolds—on a question of Beauville*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 375–403 (2013).

[LP12] V. Lazić and Th. Peternell, *On the Cone conjecture for Calabi-Yau manifolds with Picard number two*, to appear in Math. Res. Lett., arXiv:1207.3653.

[Mat14] D. Matsushita, *On almost holomorphic Lagrangian fibrations*, Math. Ann. **358** (2014), no. 3-4, 565–572.

BRILL-NOETHER METHODS IN THE STUDY OF HYPER-KÄHLER AND CALABI-YAU MANIFOLDS

FRANK GOUNELAS, MARGHERITA LELLI–CHIESA, GIOVANNI MONGARDI, GIULIA SACCÀ

Our group participated in the Junior Hausdorff Trimester Program with the project "Brill-Noether methods in the study of hyper-Kähler and Calabi-Yau manifolds". Examples of hyper-Kähler varieties are provided by Hilbert schemes of points on $K3$ surfaces and generalized Kummer manifolds. Their geometry is strictly connected with the study of line bundles on curves lying on symplectic surfaces (i.e., abelian or $K3$ surfaces), which goes under the name of Brill-Noether theory.

Research activities. A first research topic concerns curves on abelian surfaces and was carried out by Lelli-Chiesa and Mongardi together with Knutsen, who visited HIM in January. Severi varieties and Brill-Noether theory of curves on $K3$ surfaces are well understood. Quite little was known for curves on abelian surfaces and the paper "Severi varieties and Brill-Noether theory of curves on K3 surfaces" partially redresses this imbalance. Given a general abelian surface S with polarization L of type $(1, n)$, non-emptiness and regularity of the Severi variety parametrizing δ-nodal curves C in the linear system $|L|$ is proved. The gonality of the normalization $\widetilde{C}$ of C (i.e., the minimal integer k such that $\widetilde{C}$ has a $k : 1$ map to $\mathbb{P}^1$) is then studied: even in the smooth case, this is not constant when moving C in $|L|$. The second part of the paper is focused on linear series of type g^r_d with $r \geq 2$; roughly speaking, these correspond to rational maps $\varphi : C \dashrightarrow \mathbb{P}^r$ with image of degree d. It turns out that in some unexpected cases the locus $|L|^r_d$ of smooth curves in $|L|$ possessing a g^r_d is nonempty and has the expected dimension. As an application, one obtains the existence of a component of the Brill-Noether locus $M^r_{g,d}$ having the expected codimension in the moduli space of curves M_g.

In the paper "Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds", the above results are used in order to construct rational curves on Hilbert schemes of points on $K3$ surfaces and generalised Kummer manifolds. All wall divisors (i.e., divisors describing the birational geometry of these manifolds) can be obtained, up to isometry, as dual divisors to such rational curves. The locus covered by the rational curves is then described, thus exhibiting algebraically coisotropic subvarieties that deform to general small deformations of the manifold. This provides strong evidence for a conjecture by Voisin concerning the Chow ring of irreducible holomorphic symplectic manifolds.

During his stay, Mongardi made also advances in the topic of his Ph.D thesis: the paper "Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$-type" extends results about automorphisms on Hilbert schemes of two points on a $K3$ surface to Hilbert schemes of an arbitrary number of points and their deformations. This topic is deeply linked with derived autoequivalences of $K3$ surfaces.

While staying at the HIM, Frank Gounelas started interactions with Yohan Brunebarbe at the Max Planck Institute, leading over the past year and a half to three projects. The first one (joint with John Christian Ottem) is related to the geometry of Calabi-Yau varieties and consists in studying the cone of nef and effective divisors

on the projectivization of their cotangent bundle. This is still work in progress and a preprint will be posted online over the next months. Other projects include structure results for surfaces with nef cotangent bundles and positivity results for foliations.

The interaction with other groups was also important. For instance, Mongardi and Saccà collaborated with Rapagnetta in studying a particular birational morphism on a hyperkähler manifold that is deformation-equivalent to the Hilbert scheme of three points on a $K3$ surface. The quotient of the manifold by this morphism turns out to have a symplectic resolution which is deformation-equivalent to O'Grady's six dimensional example: one can use this fact in order to compute its cohomology. This project is still going on.

Organization activities. We actively participated at the trimester scientific life and organized the following activities:

- Three mini-courses held by E. Arbarello, A. L. Knutsen and E. Sernesi, who lectured about "Curves on $K3$ surfaces", "Linear series on singular curves on $K3$ surfaces: vector bundle methods and degenerations"and "Syzygies of special line bundles on curves", respectively.
- A workshop on our research topic with nine invited talks by G. Farkas, P. Frediani, M. Kemeny, G. Pacienza, G. Sankaran, J. Sawon, P. Stellari, A. Verra, C. Voisin

Papers.

(1) A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Severi Varieties and Brill-Noether theory of curves on abelian surfaces*, `arXiv:1503.04465v2`.
(2) A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds*, `arXiv:1507.06891`.
(3) G. Mongardi, *Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$-type*, `arXiv:1405.3232`, to appear in Math. Z..

Complex Projective Threefolds
Final report, September 2015

Ernesto Mistretta, Matteo Penegini,
Francesco Polizzi, Antonio Rapagnetta and Pietro Sabatino

The theme of our Research Project at HIM was the geometry of Complex Projective Threefolds. Eventually, however, the collaboration with the members of the other groups and with the many HIM visitors also produced results in areas not directly related to the main topic.

During our stay at HIM we worked together on the geometry of complex projective varieties. For instance, M. Penegini and F. Polizzi studied the existence of fibrations on irregular algebraic surfaces (see [**1**]), whereas F. Polizzi, A. Rapagnetta and P. Sabatino investigated the factoriality of threefolds with isolated singularities (see [**2**] and [**3**]).

Furthermore, the interaction of our group members with the researchers visiting HIM in occasion of the many workshops organized during the semester led to the paper [**4**] and its sequel [**5**], concerning an asymptotic bound for the number of connected components of the moduli space of surfaces of general type, and to the article [**6**], whose aim is to shed some light on the relationships among some notions of positivity for vector bundles that arose in recent decades.

As long-term HIM visitors and members of the Junior Trimester Program, we actively participated in several HIM activities. In particular:

- we organized the workshop *Geometry of complex threefolds* (March 10-13, 2014), which featured 2 minicourses (4 hours each), 4 invited talks and 32 registered participants;
- we took part in the organization of the Trimester Seminar, inviting the following people for a talk: Sonke Rollenske (Bielefeld), Marco Franciosi (Pisa), Gilberto Bini (Milano), Remke Kloosterman (Berlin), Stefano Urbinati (Padova);
- we organized two specialized seminars, held in parallel about twice a month: *Topology of singular hypersurfaces* (F. Polizzi, A. Rapagnetta and

P. Sabatino) and *Fibrations of surfaces to curves* (E. Mistretta and M. Penegini).

The HIM provided a very nice and friendly environment which helped us to pursue our research project and to cooperate in small groups. We greatly enjoyed the quiet atmosphere, the comfortable offices and the daily routine of coffee and cake. The interaction with the other groups of the Junior Trimester Program was quite interesting and stimulating. The central location of the Hausdorff Institute allowed us to have fruitful scientific exchanges with the researchers of both the University of Bonn and the Max-Planck Institute.

We are grateful to the HIM for funding our work and for giving us the opportunity of spending three productive months in the lovely city of Bonn.

References

[1] M. Penegini, F. Polizzi: *A note on surfaces with $p_g = q = 2$ and an irrational fibration*, arXiv:1407.5477 (2014). To appear in Advances in Geometry.

[2] F. Polizzi, A. Rapagnetta, P. Sabatino: *On factoriality of threefolds with isolated singularities*, Michigan Mathematical Journal **63**, Issue 4 (2014), 781-801.

[3] F. Polizzi, A. Rapagnetta, P. Sabatino: *On factoriality of threefolds with isolated singularities, II*. Work in progress.

[4] M. Lönne, M. Penegini: *On asymptotic bounds for the number of irreducible components of the moduli space of surfaces of general type*, arXiv1402.6438 (2014). To appear in Rendiconti del Circolo Matematico di Palermo.

[5] M. M. Lönne, M. Penegini: *On asymptotic bounds for the number of irreducible components of the moduli space of surfaces of general type II*, arXiv:1507.05779 (2015).

[6] T. Bauer, S. Kovács, A. Küronya, E. Mistretta, T. Szemberg, S. Urbinati: *On positivity and base loci of vector bundles*, European Journal of Mathematics **1** (2015), no. 2, 229-249.

DERIVED CATEGORIES OF HYPERKÄHLER VARIETIES FINAL REPORT, SEPTEMBER 2015

NICOLAS ADDINGTON, WILL DONOVAN, AND CIARAN MEACHAN

Project. We participated in the Junior Hausdorff Trimester Program on algebraic geometry from January to April 2014. Our project title was "Derived categories of hyperkähler varieties."

Hyperkähler or *irreducible holomorphic symplectic* varieties are one of three basic classes of varieties with vanishing first Chern class, and are the most mysterious of the three owing to the small number of examples. The best-understood examples are the "varieties of $\mathrm{K3}^{[n]}$-type," which include moduli spaces of sheaves on K3 surfaces. In recent years, several important results on varieties in this family have been obtained using *derived categories of coherent sheaves* both directly, e.g. the work of Bayer–Macrì and Bayer–Hassett–Tschinkel on the minimal model program for varieties of $\mathrm{K3}^{[n]}$-type, and indirectly, e.g. Markman's work on the monodromy group, which takes Verbitsky's Torelli theorem for general hyperkählers and gives a much more effective statement for varieties of $\mathrm{K3}^{[n]}$-type.

One of our dreams is to describe every hyperkähler variety as a moduli space of objects in a triangulated category that behaves like the derived category of a K3 surface. This might allow us to extend the results just mentioned to other families of hyperkähler varieties, or even to find new families or explain why there are so few. Since 2011, N.A. and Tom Bridgeland (Sheffield) have tried several unsuccessful approaches to constructing such a category. While in Bonn, W.D. brought in some new ideas, related to [8] below; this new approach has yet to bear fruit, but we continue to work on it.

Papers. The main product of our time together in Bonn was the following pair of joint papers:

[1] N. Addington, W. Donovan, and C. Meachan. Mukai flops and $\mathbb{P}$-twists. Preprint, 1507.02595.

Mukai flops are an important class of birational maps between hyperkähler varieties. Associated to a Mukai flop $X \dashrightarrow X'$ are on the one

hand a sequence of equivalences $D^b(X) \to D^b(X')$, due to Kawamata and Namikawa, and on the other hand a sequence of autoequivalences of $D^b(X)$, due to Huybrechts and Thomas. In this paper we work out a complete picture of the relationship between the two. We make use of these results in a particularly rich example in [2]. W.D. has proved related results for other classes of flops in work with Ed Segal and Michael Wemyss, including [8].

[2] N. Addington, W. Donovan, and C. Meachan. On derived categories of moduli spaces of torsion sheaves on K3 surfaces. Preprint, 1507.02597.

In 2011, N.A. constructed a new autoequivalence of the derived category of the Hilbert scheme of n points on a K3 surface, and conjectured that the same construction would work for any moduli space of sheaves on a K3 surface. In this paper we prove the conjecture for many moduli spaces of torsion sheaves, and find a geometrically meaningful interpretation of the autoequivalences both in the new examples and the earlier one. This paper makes use of the results of [1].

We also finished or started the following papers with other collaborators:

[3] N. Addington, W. Donovan, and E. Segal. The Pfaffian–Grassmannian equivalence revisited. *Alg. Geom.*, 2(3):332–364, 2015. 1401.3661.

This paper is unrelated to the hyperkähler project, but we put the finishing touches on it shortly after arriving in Bonn.

[4] A. Krug and C. Meachan. Spherical functors on the Kummer surface. *Nagoya Math. J.*, to appear. 1402.1651.

This note completes an earlier result of C.M. on autoequivalences of derived categories of higher-dimensional Kummer varieties, by showing that the same construction on Kummer *surfaces* yields an autoequivalence which factors into a product of familiar terms. Krug was based at Universität Bonn at the time.

[5] N. Addington and M. Lehn. On the symplectic eightfold associated to a Pfaffian cubic fourfold. *J. Reine Angew. Math.*, to appear. 1404.5657.

This note completes a result of Lehn, Lehn, Sorger, and van Straten, who constructed a new hyperkähler 8-fold but left open the question of whether it was of $\mathrm{K3}^{[4]}$-type or represented a whole new family of hyperkählers, by showing that it is of $\mathrm{K3}^{[4]}$-type. Pleasingly, derived categories appear in the proof only, not in the statement of the results,

so we are really "using derived categories to do honest geometry." We started working on this paper the previous fall, but we were able to work much faster once N.A. got to Bonn as Lehn is based in Mainz.

[6] C. Meachan and Z. Zhang. Birational geometry of singular moduli spaces of O'Grady type. Preprint, 1404.6783.

This paper follows Bayer–Macrì's work on the minimal model program for varieties of $K3^{[n]}$-type, mentioned above, proving similar results for another class of hyperkähler varieties.

[7] N. Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, to appear. 1405.4902.

The variety of lines on any cubic 4-fold is a variety of $K3^{[2]}$-type; this note describes exactly when it is birational to a Hilbert scheme of 2 points or another moduli space of sheaves on a K3 surface, conditions thought to be relevant to the question of whether the cubic is rational. Derived categories do not appear in the statements or the proofs, but they are present philosophically. This paper benefitted from N.A.'s conversations in Bonn with JHTP participants François Charles and Giovanni Mongardi.

[8] W. Donovan and M. Wemyss. Twists and braids for general 3-fold flops. Preprint, 1504.05320.

Not about hyperkählers, but like [1] it deals with equivalences and autoequivalences associated to a certain class of flops; and ideas about noncommutative deformations from this paper and its predecessor revived the project with N.A. and Tom Bridgeland mentioned earlier.

Other activities. Together with Nathan Broohmead's group ("The derived category of a T-variety"), we organized a workshop on derived categories which ran from February 10 to 13. N.A. and W.D. gave a mini-course on autoequivalences at the request of some other JHTP participants. We all attended many workshop and seminar talks by the other groups and their guests, and had many interesting mathematical conversations with them.

We found our time in Bonn very productive, and the working conditions and collegial atmosphere very pleasant. N.A.'s family stayed in Bonn for the duration of the program and have fond memories of their time there.

It is a pleasure to thank the Hausdorff Institute for its hospitality.

Final report: Derived Categories of T-varieties

- Nathan Broomhead visited the HIM for the period 06/01/2014 - 06/03/2014
- Andreas Hochenegger visited the HIM for the period 06/01/2014 - 25/04/2014
- Hendrik Süß visited the HIM for the period 09/02/2014 - 10/03/2014

Research activity

The project that we proposed was to understand the derived categories of T-varieties, ie. normal varieties X with the effective action of some torus T (see [1] for a survey). For such an X, there is a rational map $X \dashrightarrow X/T$ to its Chow quotient, and using this map, X can be described by some combinatorial object on X/T. The combinatorics describe the fibres of this quotient map, which has a toric variety Z as its general fibre.

The original aim of the project was to see if the property of being a T-variety is a derived invariant. In other words, for a (smooth projective) T-variety X and a variety Y such that $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$ we asked whether it follows automatically that Y is also a T-variety. This question has a positive answer, and the proof turned out to be far simpler than we had previously envisioned; using an application of [6, Thm. 4.18], it is rather immediate that there is also an induced action of T on Y, which makes Y a T-variety. We decided that this observation was unfortunately not worth a paper.

Given an equivalence between T-varieties, our next step was then to ask whether this equivalence also induces an equivalence of the derived categories of the Chow quotients or the (general) fibres. Even though this seems a reasonable expectation, we were able to construct some examples where the fibres and the Chow quotients are only birational but not derived equivalent. It remains an interesting question to understand which of the Fourier-Mukai kernels giving equivalences $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induce derived equivalences of the Chow quotients or the fibres. However the Chow quotient map lacks many desirable properties like flatness, and a general result in this direction seemed out of reach.

We decided therefore to change our focus in this project, and instead to look at the derived categories of a different generalisation of toric varieties, namely of Mori dream spaces, which include an important class of T-varieties, namely those of complexity one.

Given a complete normal variety, the Cox ring is defined as a k-vector space spanned by the global section of all divisor classes:

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}(D)).$$

A natural multiplication can be defined, as well. If $\mathrm{Cox}(X)$ is finitely generated as a k-algebra X is called a Mori dream space. Such a variety comes with a natural embedding into a toric variety and with a quotient representation $X = \hat{X} /\!\!/ H$ of an affine variety by a torus H. The birational geometry of X is governed by the geometric invariant theory (GIT) of this quotient. Here, T-varieties give rise to a wide class of interesting examples, since for them the Cox ring and the described quotient construction are known very explicitly [3].

In certain smooth situations Ballard, Favero and Katzarkov [2] describe how the derived category of a GIT quotient $\hat{X} /\!\!/_\chi H$ varies with the choice of character χ. As an application, they obtain an exceptional sequence for any projective toric variety X. By choosing a path in the GIT fan from the chamber corresponding to X to the boundary, they obtain a sequence of birational transformations until one arrives at a Mori fibration over a toric varity Y of lower dimension. $\mathcal{D}^b(X)$ can

be written as a semi-orthogonal decomposition whose parts come from the wall-crossings and $\mathcal{D}^b(Y)$. Both contributions come from lower dimensional varieties, so this allows an inductive procedure. This can be seen as a generalisation of the results of Kawamata ([4], [5]). One could hope for a similar description in the case of a general Mori dream space (or more precisely to the associated stack). There is however a significant obstacle: Cox rings of non-toric varieties are always singular, which prevents us from directly applying the theory of [2]. Moreover, the case of Picard rank one (which is due to Orlov), shows that the description of the derived category of X should involve contributions from singularity categories associated to the singularities of the Cox ring.

In all of the explicit examples that we have calculated so far we are able to circumvent these problems, using techniques such as root constructions, and produce semiorthogonal decompositions. However, the machinery has become complex and there are still complications in producing general statements. We expect that there is at least a reasonable class of Mori Dream stacks, where our techniques will yield interesting semi-orthogonal decompositions. For this reason we don't yet have a completed preprint. However work on the project is still ongoing. N.B. and A.H. met to discuss the project including regular meetings when A.H. visited Hannover (1/15-2/15) and when N.B. visited Köln (6/15-7/15). N.B. and H.S. have met several time to discuss, including two days in Edinburgh (5/14) and a meeting in Manchester in September 2015.

Other activities

Together with the group "Derived categories of hyperkähler varieties" we organised a workshop on derived categories which ran in the Hausdorf Institute between 10-13th February 2014. We participated in many workshops and seminars and mini-courses run by other groups. We also had many interesting discussions with members of other groups.

References

[1] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß and Robert Vollmert, *The geometry of T-varieties,* In: Piotr Pragacz (Editor) *Contributions to Algebraic Geometry*, EMS Series of Congress Reports, 17–70, 2012.

[2] Matthew Ballard, David Favero and Ludmil Katzarkov, *Variation of Geometric Invariant Theory Quotients and Derived Categories,* `arXiv:1203.6643`.

[3] Jürgen Hausen and Hendrik Süß, *The Cox ring of an algebraic variety with torus action,* Adv. Math. 225(2):977–1012, 2010.

[4] Yujiro Kawamata, *Derived categories of toric varieties,* Michigan Math. J. 54(3):517–535, 2006.

[5] Yujiro Kawamata, *Derived categories of toric varieties II,* Michigan Math. J. 62(2):353–363, 2013.

[6] Raphaël Rouquier, *Automorphismes, graduations et catégories triangulées,* J. Inst. Math. Jussieu 10(3):713–751, 2011.

Report of the group
"p-adic methods in Arakelov geometry and Shimura varieties"

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Junior Hausdorff Trimester Program

Algebraic Geometry

Period of stay: January - April 2014

1. The group

In 2013 a group of people got together and thought that it might be a very good idea to create an arena for various questions in the intersection of the theory of Shimura varieties, Arakelov geometry and p-adic geometry. These were Dennis Eriksson, Gerard Freixas i Montplet, Marc-Hubert Nicole and Siddarth Sankaran. After having been granted the scholarship, we were further enriched by Giuseppe Ancona who works on motives. All of us stayed for the entire duration of the programme, except Marc-Hubert Nicole who stayed for 3 months.

2. Organized activities at HIM

- Workshop in *Arithmetic intersection theory and Shimura varieties*.
- Mini-course on *p-adic geometry* by Jérôme Poineau.
- Mini-course on *Shimura varieties* by Marc-Hubert Nicole.
- Mini-course on *Motives* by Giuseppe Ancona.
- Organized the semi-weekly seminar PAS (12 speakers)

3. Research themes

One of the main objectives of our proposal was to gather together researchers with diverse backgrounds but common interests. While the main theme can be interpreted as arithmetic geometry, the more specific interests were in Arakelov geometry (intersection theory in the arithmetic sense, the intersections produce real numbers) and Shimura varieties (the geometric study of modular forms), with a particular view towards phenomena involving prime numbers p. The specific projects comprising our research proposal, are much motivated to some degree by Kudla's programme. Inspired by seminal work of Hirzebruch and Zagier, it is a deep conjectural framework which seeks to relate generating series of 'special' cycles on Shimura varieties with q-expansions of modular forms. Naturally, Arakelov geometry plays a key role, and most of the projects we describe either illuminates, extends, or draws analogy with, some aspect of the techniques involved in Kudla's programme. A more specific overview:

p-adic Kudla programme [N.] Inspired by Kudla's philosophy, Marc-Hubert has been investigating the arithmetic geometry of some explicit theta lifts especially in their p-adic avatars, with the hope of developing what he envisions as an emerging p-adic Kudla program. In a toy setup ("liftings from GL_2"), it replaces: Eisenstein series by explicit liftings of any classical modular form f such as the Saito-Kurokawa

lifting (or the Shintani-Shimura-Waldspurger lifting; other theta liftings, etc.) in the context of p-adic families of modular forms; and the classical derivative by the p-adic derivative of the weight varying p-adically. One can propose to view as a special case of a p-adic Kudla program a collection of rather recent results combining p-adic techniques: p-adic families of modular forms, p-adic functoriality à la Langlands and algebraic cycles over local fields.

Riemann-Roch in Arakelov geometry and Jacquet-Langlands correspondance [E., F., S.] We have been exploring the relation between two kind of results in different branches of arithmetic geometry. The first one is the Riemann-Roch formula in Arakelov geometry. The second one is the Jacquet-Langlands correspondence in the theory of automorphic forms. In their common range of application, both results are related to the trace formula. For instance, the Riemann-Roch formula involves global invariants of automorphic forms (like eigenvalues under laplacian, Hecke eigenvalues), and the Jacquet-Langlands correspondence establishes relations between such invariants, for automorphic forms for inner forms of a given reductive group. Using a conjectural version of the arithmetic Riemann-Roch formula for open varieties, in the Hilbert modular case, we show that two (arithmetic) intersection numbers on varieties related by Jacquet-Langlands correspondance should coincide. The two numbers have only recently appeared in the literature, and our computations give an positive answer to this consequence of the conjectural Riemann-Roch formula. We also independently compute various other intersection numbers.

Realizing Riemann-Roch via theta functions [E., F.] The Riemann-Roch formula, in Grothendieck's formulation, gives the equality of certain characteristic classes associated to a fibration. In the particular case of fibration of curves, it has been known since some time, that some of these classes can be lifted to natural line bundles, and that the equality can be replaced by an isomorphism. Most of the methods on the market constructing the isomorphism are either entirely non-explicit, or slightly wrong. In this project we use theta functions and discriminants to realize the isomorphism in an explicit fashion. The current plan is to publish an extended version of this in book-form, but we are still awaiting a confirmation from the editors.

Asymptotic of fiber integrals [E. , F., Christophe Mourougane] For a fibration with compact complex manifold fibers of dimension n, $X \to D^*$, there are several studies of the asymptotics of fiber integrals as one approaches singular fibers. In this project we study the problem of integrating $\eta_t \wedge \overline{\eta_t}$, where η is a relative $(n,0)$-form and let $t \to 0$ where 0 is a singular fiber. We relate these asymptotic formulas with well-known invariants from the mixed Hodge structure on the general fiber, and apply it to find analytical approaches to Kodaira-type canonical bundle formulas.

Pure motives attached to modular forms [A.] Scholl constructed motives associated with modular forms, lifting Deligne's construction of Galois representations attached to modular forms. This is done by studying the cohomology and the motive of modular curves (and the universal family of elliptic curves over them). A program, initiated by Wildeshaus, aim to generalize such constructions to all PEL Shimura varieties.

The Galois representations constructed by Deligne (as well as the motives constructed by Scholl) are *pure*. This is essential, for exemple, for Deligne's proof of Ramanujan's conjecture on the τ-function. On the other hand these modular forms live in the cohomology of varieties that although smooth are not compact (such as

modular curves). Hence to understand purity one has to deal with *boundary cohomology*. In a preprint we settle some new cases.

Written works:

As a consequence of the 10h lecture series on Shimura varieties that N. gave at the H.I.M. in January 2014, he wrote a book chapter on the classical theory of unitary Shimura varieties to appear in the "Formes automorphes" book project directed by Michael Harris (IMJ, Columbia Univ.):

- Nicole, M.-H., Unitary Shimura Varieties, book chapter to appear in *Shimura Varieties and their associated Galois Representations* , Paris book project, vol. II, preprint, 25 pages, Oct. 2014.

- Giuseppe Ancona, *Degenerations of Hodge structures over Picard modular surfaces*, 15 pages, submitted.

- The other works mentioned are to appear in preprint form in a near future.

4. Working conditions

The set-up of the Institute made possible 10h long undisturbed working days. Indeed, the now famous 4 pm Institute cake would allow us not only to mingle and socialize with other participants, but also to get easy calories intake to last till a late supper, therefore adding 2-3h of work time to the day. On a related note, it was also very convenient to have simple access to bikes and discounts on several gyms in town.

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