Report on the project 'Intrinsic meaning of local gauge invariance' at the Hausdorff Research Institute for Mathematics, Bonn

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1 Research subjects, new results, future directions

The group consisted of four researchers: Katarzyna Rejzner, Jan Schlemmer, Yoh Tanimoto and Wojciech Dybalski. During our stay at the HIM we discussed and worked on the following questions related to the problem of local gauge invariance (among others):

1. *Classical Electrodynamics and directional charges.* At the heuristic level, directional charges arise from local gauge symmetry via the Noether theorem. (Similarly to the electric charge, which arises from global gauge symmetry). We established their existence rigorously in a system describing a charged particle interacting with the electromagnetic field. Moreover, it turned out that the directional charges are a promising tool for asymptotic analysis of solutions of the Maxwell equations – a fact which apparently escaped attention so far. Heuristically, their conservation enforces slow, Coulomb-like decay of the radiation emitted by an accelerating charge. This observation should allow further development of scattering theory of Maxwell equations coupled to charged particles presented in [9].

2. *Quantum Electrodynamics and infraparticle problem.* It is well-known that a charged particle does not have a sharp mass in generic representations of Quantum Electrodynamics as it is always accompanied by soft photons. It has, however, been recently conjectured by Buchholz and Roberts that there exist special charged representations where this problem disappears. We have constructed such representations in the framework of non-relativistic Quantum Electrodynamics. It turned out that they have rather exotic features: Neither the position operator of the charged particle nor the asymptotic photon fields exist, which invalidates the conventional scattering theory.

3. *Non-relativistic Quantum Electrodynamics and directional charges.* It is well known that quantization of gauge theories requires an (arbitrary) choice of gauge to eliminate superfluous degrees of freedom. The question if different choices lead to the same quantum theory appears to be unanswered to date. Our conjecture is that this is not the case: Different quantization prescriptions may give rise to different values of directional charges. A convenient framework to verify this conjecture is that of non-relativistic Quantum Electrodynamics. We expect that its

quantization in the Coulomb gauge and in an axial gauge will give rise to different values of directional charges and therefore to different quantum-mechanical theories. While quantization in the Coulomb gauge is well known [9], quantization of non-relativistic Quantum Electrodynamics in axial gauges remains to be performed, e.g. following [7].

Apart from the problem of local gauge invariance we worked on several other aspects of quantum field theory. Papers completed during our stay at the HIM concern the problem of asymptotic completeness [3, 2] and construction of quantum field theories [6, 10].

2 Organization of HIM activities

1. *Workshop 'Algebraic Quantum Field Theory and Local Symmetries'*, 26.09.2012–28.09.2012. The workshop gave an overview of recent developments toward mathematical understanding of the problem of local gauge invariance. The speakers include K.Fredenhagen and R.Longo.

2. *AQFT seminar*. From October 2012 we organized a series of talks almost on a weekly basis on AQFT and related fields. The speakers include three of us (W.D., J.S., Y.T.) and A.Pizzo. Several members of other groups and some researchers in Bonn attended our seminars.

3. *Trimester Seminar*. Jointly with other groups we orgnized a weekly seminar. Three of us (K.R., J.S., Y.T.) and J.Zahn gave talks.

3 Later work

After the end of the program the group members published several papers with an acknowledgement of their stay at the HIM. These works concern the infrared problem in non-relativistic QED [4, 5], local gauge invariance in perturbative relativistic QFT [8], construction of QFT models [11, 1] and the problem of non-interaction in conformal field theory [12].

4 Conclusion

It is clear from the preceding sections that the HIM offers an excellent environment for scientific activity. We enjoyed our stay at HIM and we are looking forward to future events at the HIM in the field of Mathematical Physics.

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Junior Hausdorff Trimester Program

Mathematical Physics

Report on the activities of Group D - "BPS states"

Sven Meinhardt and Jacopo Stoppa

The activities of Group D "BPS States" did not focus on a single open problem, but rather reflected the different interests of its members (both physicists and mathematicians): Nils Carqueville, Ben Davison, Mario Garcia-Fernandez, Stefan Hohenegger, Jan Manschot, Sven Meinhardt, Jacopo Stoppa, Tom Sutherland.

The common theme was provided by the concept of a Bogomolny-Prasad-Sommerfield state in a physical theory (which arose first in the study of certain gauge theories) and its different incarnations in mathematical physics, algebraic geometry and differential geometry.

More specifically, the algebra, geometry and physics of BPS states were investigated in the contexts of topological field theory (Carqueville), supersymmetric gauge and string theories (Hohenegger, Manschot), Donaldson-Thomas theory (Davison, Meinhardt, Stoppa), cohomological Hall algebras (Davison, Meinhardt), geometric aspects of stability conditions on categories (Garcia-Fernandez, Stoppa, Sutherland).

Altogether the activities of the group led to 7 publications, which are listed as preprints **2012c06**, **2012c07**, **2012c08**, **2012c10** (now part of arXiv:1403.7404 [math.AG]), **2012c12**, **2012c15**, **2012c16** in the Trimester publications page; the same page will be kept up to date with publication history.

Apart from personal collaborations, the group activities included numerous informal seminars devoted to both basic and advanced topics in the physics and geometry of BPS states, as well as a colloquium style, monthly Trimester Seminar. In the course of the Trimester, Group D hosted numerous short-term visitors (a complete list can be found on the Trimester webpage), and ran two more formal mini-courses. Additionally we organised a workshop on the algebra, geometry a physics of BPS states (November 12-14, 2012), involving some 20 participants. Members of the Group also took part in an outreach initiative for the HIM; a podcast was produced and is available on the HIM Junior Trimester Program description page.

We feel very privileged to have had this opportunity to work at the HIM. The JHRT has been a very exciting, productive, and pleasant experience, which we believe will influence the research of the group members for many years to come.

Finally we like to mention that most members of Group D reconvened for the follow-up workshop "Donaldson-Thomas invariants, BPS states and related topics", Pavia, 25 - 27 June 2013.

Junior Trimester Program "Mathematical Physics" (September - December 2012)

Final Report

Conformal field theory and Moonshine (Group A)

Thomas Creutzig and Gerald Höhn

The Project

Our collaboration was centered around two interrelated projects: The structure theory of vertex and generalized Kac-Moody algebras and its interplay with Mathieu moonshine.

Mathieu moonshine started with the observation of Eguchi, Ooguri and Tachikawa in 2010 that there is a weakly holomorphic mock modular form of weight 1/2, called $H(\tau)$, whose coefficients as a power series coincide with dimensions of representations of the largest Mathieu group M_{24}. This function is naturally associated to the weak Jacobi form of weight zero and index one, the universal complex elliptic genus of $K3$ surfaces. Sigma models on $K3$ surfaces possess the $\mathcal{N} = 4$ super Virasoro vertex algebra of central charge $c = 6$ as symmetry and this weak Jacobi form can be decomposed into a sum of super characters of modules for this super algebra with $H(\tau)$ counting the multiplicities. Explicit McKay-Thompson series for the conjectured M_{24}-module of Eguchi, Ooguri and Tachikawa have been proposed and then used by Gannon to establish numerically the existence of such a module. The open problem is to find a natural and mathematical rigorous defined structure on which M_{24} (or large subgroups) act by automorphisms explaining the occurrence of this M_{24}-module.

Our work during the activity at the Hausdorff Institute resulted in the two joined papers [CHM] and [CH] on Mathieu moonshine. In addition, Creutzig invited Simon Wood (IPMU Tokyo) and David Ridout (ANU Canberra) for a week in December continuing work on vertex algebras which led to the publications [CR, CRW]. Höhn started calculations related to the automorphism groups of CFTs of K3 surfaces which resulted in a joint paper with Geoffrey Mason (UC Santa Cruz) on the classification of symplectic automorphism groups associated to the second Hilbert scheme of K3 surfaces [HM].

In the note [CHM] (jointly with T. Miezaki, Yamagata University) we describe the parity of the coefficients of the McKay-Thompson series of Mathieu moonshine by using Sturm's theorem on congruences of coefficients of modular forms. As an application, we prove a conjecture of Cheng, Duncan and Harvey stated in connection with Umbral moonshine for the case of Mathieu moonshine which predicts the appearance of certain irreducible M_{24}-representations in the M_{24} moonshine module.

In the paper [CH], we relate the Mathieu group M_{24} and the complex elliptic genus of a K3 surface with the symmetries of geometric structures on K3 surfaces. Physicists have studied the sigma model on K3 surfaces resulting in a moduli space of CFTs. However, not all the postulated properties of a generic CFT from that moduli space are yet verified mathematically rigorously. I our paper, we avoid any implicit assumptions. It is a well-known result of Mukai that a finite group of symplectic automorphisms of a $K3$ surface must be contained in a subgroup of M_{23} with at least five orbits on its natural presentation on 24 points. Our first result shows that a virtual vector space which is (in a compatible way) a representation for all groups allowed by Mukai's theorem has also the structure of a virtual M_{24} representation with the given representations of the Mukai groups as the restrictions. It follows that the complex elliptic genus of a K3 surface is a virtual module for the Mathieu group M_{24} compatible with the symplectic groups actions. Our second result shows that the topologically defined elliptic genus of a K3 surface is the graded character of a virtual module of a vertex algebra $V^{\mathrm{SU}(2)}$ called the holonomy subalgebra which contains the $\mathcal{N}=4$ super Virasoro vertex algebra of central charge $c=6$ as a subalgebra. We also find explicit formulas for the multiplicities of the decomposition. Finally, we prove that the equivariant topological elliptic genus for those conjugacy classes in M_{24} which arise from a symplectic action coincides with McKay-Thompson series of the proposed M_{24} moonshine module.

The orbifold second quantized elliptic genus, i.e. the orbifold elliptic genus for the symmetric powers of $K3$, coincides by a theorem of Borisov and Libgober with the elliptic genus of the corresponding Hilbert schemes. The formula for the second quantized elliptic genus formula can also be considered equivariantly similar to the formulas obtained by Borcherds in the Monstrous moonshine case. These formulas allow to compare the prediction of Mathieu moonshine with the equivariant elliptic genus of finite symplectic automorphisms of Hyperkähler manifolds deformation equivalent to a Hilbert scheme of a $K3$ surface. In particular, it is known that there are additional finite symplectic automorphisms not induced from symplectic automorphisms of a $K3$ surfaces (sometimes called non-natural automorphisms). In the paper [HM] by Höhn and Mason, all classes of finite symplectic automorphisms on Hyperkähler manifolds deformation equivalent to the second Hilbert scheme of a $K3$ surface are determined. This generalizes work by Mukai, Xiao, Kondo and Hashimoto for K3 surfaces and by Mongardi and Huybrechts for $K3^{[2]}$.

Our stay at the Hausdorff Institute

Our group invited the following speakers for the weekly Trimester Seminar: Nils Scheithauer (Darmstadt), Miranda Cheng (Paris) and David Ridout (Canberra). In addition, we invited Simon Wood (Tokyo) and Simon Norton (Cambridge, UK) for short visits in December for discussions related to our project.

We had intensive discussions with members of the group "BPS states" (Group D) which was the one most closely related to our project, but we also enjoyed interactions with the two other groups.

The trimester was filled with many activities including three workshops, three mini courses and a lecture series by Gregory Moore. During the calmer days we profited from the pleasant working atmosphere at the HIM.

The paper [HM] was motivated by discussions with Daniel Huybrechts at the beginning of the program.

References

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Final Report

Geometry of Gauged Vortices

Group C – JHTP Mathematical Physics, HIM 2012

Nuno M. Romão (University of Göttingen)

1 The group

We applied to the JHTP as a group of ten researchers — Dennis Eriksson, Eduardo González, Lotte Hollands, Ignasi Mundet, Timothy Nguyen, Andreas Ott, Nuno Romão, Martin Speight, Christian Wegner and Fabian Ziltener — with the general aim of studying various aspects of the vortex equations in fibre bundles over Riemann surfaces. Gauge theory has had a fruitful interaction with many branches of mathematics, and this was well reflected in the broad spectrum of our backgrounds. To a large extent, we accredit many of the successes described in this report to the interdisciplinarity advocated in our original proposal.

2 Research projects

Our projects revolved around three main themes: **(i)** geometry and topology of vortex moduli spaces, relating to the Kähler metrics supported by vortex moduli, as well as the geometric and topological structures emerging from their quantisation; **(ii)** gauged Gromov–Witten invariants, which yield a generalisation of the theory of pseudoholomorphic curves in symplectic manifolds to the setting of holomorphic Hamiltonian actions; **(iii)** vortices and higher-dimensional field theories.

(i) Geometry and topology of vortex moduli spaces

Let G be a compact Lie group, for which a holomorphic Hamiltonian action on a Kähler manifold (X, J, ω_X) with moment map μ is given. The vortex equations are the PDEs

$$\bar{\partial}_A^{j_\Sigma, J} u = 0, \qquad F_A + \mu(u)\, \omega_\Sigma = 0$$

for pairs (A, u), where A is a connection on a principal G-bundle $P \to \Sigma$ over a surface $(\Sigma, j_\Sigma, \omega_\Sigma)$ on which another Kähler structure is given, and $u : P \to X$ is a G-equivariant map. Each u determines a G-equivariant 2-homology class $[u] \in H_2^G(X;\mathbb{Z})$, and fixing such a class $\mathbf{k}$ one obtains a moduli space $\mathcal{M}_\mathbf{k}$ of vortex solutions modulo gauge equivalence. The natural flat L^2-metric on the space of all pairs (A, u) descends to a nontrivial Kähler metric g_{L^2} on $\mathcal{M}_\mathbf{k}$, defining what we call the L^2-geometry of the moduli. In the project [25], the metrics g_{L^2} were investigated for $\Sigma = \mathbb{C}$ and $\Sigma = \mathbb{P}^1$ with the standard action of $G = \mathrm{U}(1)$ on $\mathbb{P}^1$. Combining a localisation formula with a heuristic description of these metrics as a limit of the L^2-geometry associated with the linear action of $G = \mathrm{U}(1)^2$ on $\mathbb{C}^2$, Nuno Romão and Martin Speight obtained formulas for the symplectic volume of the moduli space in the case $\Sigma = \mathbb{P}^1$ and $\mathbf{k} = (1, 1)$, which turns out to be finite. Numerical work on the PDEs has not only yielded further support for this prediction, which was expected in analogy with results on moduli of ungauged sigma-models, but it also uncovered that the vortex metrics g_{L^2} are complete, in striking contrast with the ungauged models.

Both of these properties of the L^2-geometry (completeness and finite volume) are necessary to validate a very compelling semiclassical picture for the canonical quantisation of the A-twisted supersymmetric sigma-models associated to the vortex equations above. These lead to topological quantum field theories whose correlators localise to integrals over the moduli spaces $\mathcal{M}_\mathbf{k}$. In the project [8], the quantum ground states of these gauged A-models are studied for compact Σ via $N = (2, 2)$ supersymmetric quantum mechanics on $\mathcal{M}_\mathbf{k}$; X is taken to be a symplectic compact toric manifold with torus G. In this setting, it is most natural to allow the waveforms to take values in local systems over the moduli spaces — equivalently, one lifts the supersymmetric quantum mechanics to the universal covers $\widetilde{\mathcal{M}_{\mathbf{k}}}$ with L^2-geometry induced by pull-back. The easier situation where $X = \mathbb{C}$ with usual circle action can be understood rather directly from results on the topology of $\widetilde{\mathrm{Sym}}^k\Sigma$ obtained in [5]: if Σ has genus $g > 1$, and for all $k \geq 1$, one must deal with a torus $\mathrm{Hom}(H_1(\pi_1(\mathcal{M}_k), \mathrm{U}(1)) \cong \mathrm{U}(1)^{2g}$ parametrising the possible choices of rank-one local systems over $\mathcal{M}_k = \mathrm{Sym}^k\Sigma$, interpreted as dual (electric) Aharonov–Bohm charges in Σ itself, and we showed that the L^2-cohomology $H^j_{(2)}(\widetilde{\mathrm{Sym}}^k\Sigma, \mathbb{C})$ is infinitely generated precisely in the middle degree $j = k$. To obtain sensible information about the action of $\pi_1(\mathcal{M}_k) \cong H_1(\Sigma, \mathbb{Z})$, one needs to extract L^2-Betti numbers, which is not hard in this co-compact example. The result shows that vortices in line bundles (in their ground states) are fermionic particles; moreover, Bökstedt and Romão were able to obtain a very intuitive interpretation of this result based on a 'lemniscate principle' [7] for pair-of-pants decompositions of Σ. As an application, a complete discussion of the spectrum of quantum particles has already been obtained by Romão and Wegner [8] for the rank-one ground states of the simplest nonlinear model $X = \mathbb{P}^1$: alongside the fermionic states associated to the two fixed points of the circle action, we detected the existence of a bosonic particle arising from the fusion of two fermions of each type. Moreover, the fundamental groups of the moduli spaces were computed in [6] using generalised braids that we call 'divisor links' on Σ. This calculation relied on the construction of invariants for divisor links, and it shows that the centres of $\pi_1(\mathcal{M}_{\mathbf{k}})$ are nontrivial abelian groups with possibly nonzero rank; their torsion is determined arithmetically from the classes in $H_2^G(X, \mathbb{Z})$. Conceptually, one of the most startling outcomes of this project is the realisation that abelian gauge theories may give rise to nonabelionic particles at the quantum level. The presence of such 'nonabelions' potentiates applications of abelian gauged nonlinear sigma-models in areas of recent interest in condensed matter physics such as topological quantum computation.

Another viewpoint on quantising vortices consists of regarding the moduli spaces $\mathcal{M}_{\mathbf{k}}$ themselves as phase spaces, and then proceed to the geometric quantisation of the symplectic structures ω_{L^2} in the complex polarisation induced by j_Σ. The project [11] by Dennis Eriksson and Nuno Romão studies this problem for the case of vortices in line bundles using Quillen determinants. The quantum Hilbert spaces were calculated from a choice of prequantisation of the Kähler structure $(\Sigma, \frac{\tau}{2}\omega_\Sigma, j_\Sigma)$ on the surface, where $\tau > 4\pi k/\mathrm{Vol}(\Sigma)$ reflects the choice of moment map μ. There is a convenient way of incorporating metaplectic corrections and model the different polarisations on the family $\mathrm{Pic}^k\Sigma$. It was found that there is a special case (corresponding to a metric of constant scalar curvature on Σ) where the choice of quantisation of Σ is immaterial, and one can naturally ask whether this situation also leads to special properties of the L^2-geometry. One of the main results in [11] was the proof that, whenever $g > 1$, there does not exist a projectively flat connection over $\mathrm{Pic}^k\Sigma$ relating Hilbert spaces for different holomorphic structures. There were also developments beyond previous attempts to understand localisation formulas for the prequantum connection, refining the already well-established formulas for its curvature ω_{L^2}.

(ii) Gauged Gromov–Witten invariants

A new idea in gauged Gromov–Witten theory is to incorporate the complex structure j_Σ as part of the moduli problem; a natural question to ask then is what happens when the complex curve Σ degenerates and becomes singular. The simplest example is of course the degeneration of a rational curve when a node is created. Here j_Σ remains constant up to diffeomorphism as long as Σ remains smooth, but nonetheless this example can be quite useful to explore the new phenomena related to the appearance of nodes — in fact, the need to allow for nodes in the compactification is the only feature that makes the inclusion of j_Σ in the moduli problem a highly nontrivial task. This was the starting point of a project by Ignasi Mundet; the forthcoming paper [21] studies the behaviour of the Hitchin–Kobayashi correspondence for $\Sigma \cong \mathbb{P}^1$ and X any Kähler manifold, for a family of area forms $\{\omega_{\Sigma,t}\}_t$ induced by restricting the Fubini–Study symplectic form on $\mathbb{P}^2$ to the quadrics $xy = tz^2$ (parametrised by $\mathbb{P}^1$) and letting $t \to 0$. Applying the Hitchin–Kobayashi correspondence to a family of pairs $\{(A_t, u_t) : \bar{\partial}^{j,J}_{A_t} u_t = 0\}_t$ on a fixed bundle over $\mathbb{P}^1$ and depending on t (note that the latter is an essential ingredient in the Hitchin–Kobayashi correspondence for general holomorphic pairs), one obtains a one-parameter family of vortices. Their possible limits as

$t \to 0$ are studied in algebraic terms from the point of view of a compactness theorem established by the author and Tian.

Andreas Ott and Fabian Ziltener made progress on a long-term project [24] that aims at generalising the existing definitions of gauged Gromov–Witten invariants (which so far have required rather restrictive assumptions) to monotone Hamiltonian manifolds X, including important examples such as Grassmannians and toric varieties. This new approach is based on a nonlocal version of the vortex equations (of integro-differential type) which relies on a certain holonomy perturbation $\bar{\partial}_{A,\Theta}^{j_\Sigma,J}$ of the holomorphic structure under a classifying map Θ for the gauge group action on the space of pairs (A, u). A novelty is that J now gets replaced by an equivariant family of not necessarily G-invariant almost complex structures on X, with an additional nonlocal dependence on the pair (A, u) itself. In this way, transversality for the sphere bubbles in the compactification of the moduli space can be achieved as long as X is assumed to be monotone. An axiomatic characterisation of such holonomy perturbations Θ was obtained. An important step in the whole programme is a certain a priori estimate that requires a careful study of the first and second derivatives of Θ. Ott and Ziltener found a way of constructing suitable maps Θ explicitly by taking the holonomy of the connection A along certain paths on Σ and then averaging over a compact family. This provides a conceptually clear framework for the analysis required in the generalisation of gauged Gromov–Witten invariants.

Another extension of the moduli space of vortices was the subject of a project [12] by Eduardo González, where parabolic structures are added at finitely many smooth points of Σ; the main motivation is to relate the invariants to those for gauged maps on stacky curves (orbifolds). The first step is to understand gauged maps whose principal component is a Galois quotient from a smooth curve by a finite group. The relevant semistability condition for these objects is a version of Mundet semistability supplemented by weights at the parabolic points. Progress has now been made on understanding the algebraic structure of the corresponding invariants. The usual Gromov–Witten invariants yield a cohomological field theory in the sense of Kontsevich–Manin, since the moduli space is a stack over the Deligne–Mumford stack $\mathcal{M}_{n,g}$ of stable curves; similarly, gauged Gromov–Witten invariants form a cohomological field theory trace, since the moduli spaces are now defined over the Fulton–McPherson space $\mathcal{M}_{n,g}(\Sigma)$ of stable curves with a principal component Σ by results of Woodward, and this can be extended to the parabolic case. A similar problem is treated with a slightly different perspective in ongoing work [14] on vortices over cylindrical ends; here, the main goal was to establish the existence of a virtual fundamental class for the moduli, which is rendered difficult by the presence of symmetries.

One area where gauged Gromov–Witten theory has been finding applications is mirror symmetry. Vortices (or gauged maps) interpolate equivariant cohomology with the quantum cohomology of a GIT quotient via the so-called Kirwan map. The effect of varying this quotient on the invariants was studied in the paper [15], which was revised by the first author during his stay at HIM. In another project [13] of Eduardo González with Hiroshi Iritani, the Kirwan map was identified with objects called 'Seidel elements' in the setup of toric mirror symmetry.

(iii) Vortices and higher-dimensional field theories

The vortex equations appear in higher-dimensional field theory when one introduces certain nonlocal sources (or defects). Particularly interesting are four-dimensional supersymmetric gauge theories, in which the vortex equations govern two-dimensional surface defects. One of the interesting questions is to classify such surface defects and figure out their intrinsic two-dimensional description, for example in terms of gauged sigma-models. In collaboration work [9] where Lotte Hollands took part, it was found that surface defects in an important class of four-dimensional field theories, indexed by a Lie algebra, are labeled by an irreducible representation of this Lie algebra. In two extreme cases, it was possible to obtain the corresponding two-dimensional description in terms of a gauged linear sigma-model, and to associate a new invariant to it: a two-dimensional superconformal index. Perhaps most interesting is that this invariant can be phrased as a certain elliptic difference operator, and that the kinematics of the surface defects gives rise to an algebra of such operators.

At the quantum level, vortices in supersymmetric theories provide examples of what are called BPS states: representations of a supersymmetry algebra which saturate a mass bound. There has been much interest in developing tools to understand phenomena we already encounter in the vortex setting, like wall-crossing, in higher-dimensional theories. One insightful way to study wall-crossing of BPS states in certain supersymmetric four-dimensional gauge theories is through so-called 'spectral networks', which Greg Moore told us about in his inspiring Felix Klein Lectures (hosted by HCM at the time of the JHTP). A spectral network consists of a collection of trajectories on a Riemann surface S that obey various rules; one set of examples is given by foliations of S that are generated by meromorphic quadratic differentials. Together with Andrew Neitzke, Lotte Hollands studied a more mathematical application of such spectral networks [17]. Given a spectral network $\mathcal{W}$, one can define a one-to-one mapping from nonabelian flat connections on S to abelian flat connections on a ramified cover $\tilde{S} \to S$ (a spectral curve). Particularly interesting in that this naturally defines a Darboux coordinate system on the moduli space of nonabelian flat connections on S for every choice of $\mathcal{W}$. Known coordinate systems follow from particular choices; the paper [17] discusses how to obtain the Fenchel–Nielsen coordinates, as well as the Fock–Goncharov coordinates.

3 Activities

A first meeting of the group members was organised at the occasion of the

- **Master Class:** "Vortex Equations and Hamiltonian GW-Invariants", 17th–20th January 2012 (by Ignasi Mundet), 16 lectures [22] at the Centre for Quantum Geometry of Moduli Spaces (QGM), Aarhus University, Denmark

for which we received very generous support from QGM.

In Bonn, we structured our research activities around three events that were open to everyone, and which we took care to advertise at the local mathematical institutions (HCM, HIM and MPIM):

- **Trimester Seminar:** This was run weekly in coordination with the other three groups at the JHTP; we contributed with four talks on vortices at pedagogical level by members of the group;
- **Vortex Seminar:** Meeting every week, this was intended mainly as a forum to discuss informally recent research by our visitors. A total of twelve talks were presented, in addition to a **minicourse** on "Vortices and Quantum Kirwan Maps" by Fabian Ziltener, which spanned two days;
- **Workshop** "Geometry of the Vortex Equations", 27th–30th November 2012**:** This was the main event in the whole programme, and consisted of three days dedicated to each of the central topics in our proposal, plus an extra day with a more interdisciplinary flavour. Nineteen talks were presented altogether, and they are all documented (through slides and notes) on the HIM webpage; a total of 45 participants registered. The social apex of this meeting was a memorable **recital** by and for vortex researchers, featuring vocal and piano performances by virtuosi among our group.

It should be mentioned that some of the researchers we brought to HIM from abroad made themselves available to give seminars about their work at other locations in Germany; among the institutions that profited from us in this way were the MPIM in Bonn and the Universities of Augsburg, Bielefeld, Duisburg-Essen, Heidelberg and Münster, as well as group D in our JHTP.

To ensure continuity and give visibility to the research we completed for the JHTP, we set up two further workshops at the Simons Center for Geometry and Physics, Stony Brook University, USA:

- **Workshop 1:** "Equivariant Gromov–Witten Theory and Applications" 12th–16th May 2014 (organisers: Eduardo González and Chris Woodward), in topics related to theme **(ii)** and mirror symmetry;
- **Workshop 2:** "Gauged Sigma-Models in Two Dimensions" 3rd–7th November 2014 (organisers: Sergei Gukov, Nuno Romão and Samson Shatashvili), in topics related to themes **(i)** and **(iii)**.

4 Our visitors

At the time of the JHTP, we were able to invite a total of 22 external visitors to our group, either as speakers in the weekly Vortex Seminar or participants of our workshop. In a limited amount of space, we can only mention a few highlights of such visits and of their impact on further research.

Nick Manton was present at the opening of the JHTP, and he later reported on experiments [10] aimed at understanding the nonlocal geometric flows for metrics on surfaces defined by the L^2-geometry of 1-vortices with varying size. Tudor Dimofte spent three weeks at HIM within September, and he gave an account of a project [4] on holomorphic blocks in 3-dimensional gauge theories; his fascinating talk illustrated far-reaching applications of the 'vortex counting' introduced to us by Lotte Hollands in this initial period.

In October, we had a rather short visit by Amihay Hanany, who presented his viewpoint [16] on the B-twisted gauged sigma-models associated to linear nonabelian vortex equations; supersymmetric QFTs of this type are alternative to those mentioned in **(i)**. Markus Szymik gave an account of his attempt [28] at constructing co-homotopy invariants via the vortex equations, in analogy with ideas that have led to successful applications for the Seiberg–Witten equations. João Baptista payed us a visit to provide further insight on his study of the L^2-geometry of ungauged nonlinear sigma-models using linear vortex metrics, and to describe his new results [3] on vortices in simply connected manifolds and in abelian varieties. Tim Perutz also spent a week in Bonn to discuss with Tim Nguyen ramifications of a project that stems from Nguyen's PhD thesis [23]. Developing the ideas discussed during his visit, Perutz is now working on a project with graduate supervisee Andrew Lee to devise an analogue of Heegaard–Floer theory for 3-manifolds. This theory will follow the same basic pattern as Heegaard–Floer theory, but will be based not on symmetric products of a Riemann surface Σ (alias vortices on Σ with $G = \mathrm{U}(1)$), but rather on stable (i.e. Bradlow) pairs on Σ (alias vortices with $G = \mathrm{SU}(2)$); the visit to HIM was useful in getting this project off the ground.

As a warm-up for the workshop, we had a row of lectures on gauged Gromov–Witten theory in the algebraic setting by Eduardo González as well as Bumsig Kim, who visited HIM with his graduate student Jeongseok Oh. Kim explained his theory of quasimaps in joint work with Ciocan-Fontanine, which provides a powerful framework to discuss vortex moduli in higher rank for affine targets; while in Bonn, he got inspiration for an extension of this setup [18] that should lead to interesting applications. The workshop itself was meant to be a period of congregation of vortex researchers from all over the world, some of them visiting our group for the second time. The three keynote speakers (Steve Bradlow, Chris Woodward and Sergei Gukov) gave extremely inspiring talks that covered broad ground in each of the three themes in our proposal. Around the time of his visit to take part in the workshop, João Baptista profited from various discussions to develop his new idea of looking at vortices of a given charge on a smooth surface as vortices of a lower charge in a singular background metric [2], whereas Óscar García-Prada made progress on a new project about gravitating vortices [1]. Sushmita Venugopalan and Chris Woodward were able to complete their work on classification of affine vortices [29] at the time of their visit; after the workshop, she provided us with details about her progress on vortices in noncompact surfaces, also benefiting from discussions with Fabian Ziltener.

5 Publications and preprints

The papers listed here are either by group members and connected to the projects described in section 2, or they result from research done by our visitors which has intertwined with our activities at HIM. Author names are written in **boldface** for group members, and underlined for visitors.

[1] L. Álvarez-Cónsul, M. García Fernandez, Ó. García-Prada: *Gravitating vortices* (in preparation)
[2] J.M. Baptista: *Vortices as degenerate metrics*, Lett. Math. Phys. **104** (2014) 731–747
[3] J.M. Baptista: *Moduli spaces of abelian vortices on Kähler manifolds*; `arXiv:1211.0012`

[4] C. Beem, T. Dimofte, S. Pasquetti: *Holomorphic blocks in three dimensions*; `arXiv:1211.1986`
[5] M. Bökstedt, **N.M. Romão**: *On the curvature of vortex moduli spaces*, Math. Z. **277** (2014) 549–573
[6] M. Bökstedt, **N.M. Romão**: *Divisor links and fundamental groups of toric vortex moduli* (in preparation)
[7] M. Bökstedt, **N.M. Romão**: *Pairs-of-pants, lemniscates and L^2-invariants* (in preparation)
[8] M. Bökstedt, **N.M. Romão**, **C. Wegner**: *L^2-invariants and supersymmetric quantum mechanics on vortex moduli spaces* (in preparation)
[9] M. Bullimore, M. Fluder, **L. Hollands**, P. Richmond: *The superconformal index and an elliptic algebra of surface defects*; `arXiv:1401.3379`
[10] D. Dorigoni, M. Dunajski, N.S. Manton: *Vortex motion on surfaces of small curvature*, Ann. Phys. **339** (2013) 570–587
[11] **D. Eriksson**, **N.M. Romão**: *Kähler quantisation of vortex moduli* (in preparation)
[12] **E. González**: *Parabolic gauged maps* (in preparation)
[13] **E. González**, H. Iritani: *Seidel elements and potential functions of holomorphic disc counting*; `arXiv:1301.5454`
[14] **E. González**, **A. Ott**, C. Woodward, **F. Ziltener**: *Symplectic vortices with fixed holonomy at infinity* (in preparation)
[15] **E. González**, C. Woodward: *A wall-crossing formula for Gromov-Witten invariants under variation of GIT quotient*; `arXiv:1208.1727`
[16] A. Hanany, R.-K. Seong: *Hilbert series and moduli spaces of k $\mathrm{U}(N)$ vortices*; `arXiv:1403.4950` (submitted to JHEP)
[17] **L. Hollands**, A. Neitzke: *Spectral networks and Fenchel–Nielsen coordinates*; `arXiv:1312.2979`
[18] B. Kim, J. Oh: *Quasimaps for fibrations* (in preparation)
[19] A. Lee, T. Perutz: *Floer theory in spaces of rank 2 stable pairs* (work in progress)
[20] N.S. Manton: *Vortex solutions of the Popov equations*, J. Phys. A: Math. Theor. **46** (2013) 145402
[21] **I. Mundet i Riera**: *Hitchin–Kobayashi correspondence for twisted holomorphic maps on nearly singular conics* (in preparation)
[22] **I. Mundet i Riera**: *Vortex Equations and Hamiltonian GW-Invariants* (in preparation; to be published as a book in the QGM Master Class Series, European Mathematical Society)
[23] **T. Nguyen**: *Lagrangian correspondences and Donaldson's TQFT construction of the Seiberg-Witten invariants of 3-manifolds*, Alg. Geom. Top. **14** (2014) 863–923
[24] **A. Ott**, **F. Ziltener**: *Gauged Gromov–Witten invariants for monotone symplectic manifolds* (in preparation)
[25] **N.M. Romão**, **J.M. Speight**: *L^2-geometry in gauged nonlinear sigma-models* (in preparation)
[26] **N.M. Romão**, **C. Wegner**: *L^2-Betti numbers and particle counting in a gauged nonlinear sigma-model* (in preparation)
[27] **J.M. Speight**: *Solitons on tori and soliton crystals*, Commun. Math. Phys. (to appear); DOI 10.1007/s00220-014-2104-z
[28] M. Szymik: *The stable homotopy theory of vortices on Riemann surfaces*; `arXiv:1310.7737`
[29] S. Venugopalan, C. Woodward: *Classification of affine vortices*; `arXiv:1301.7052`