Report on the Hausdorff Trimester Programme

# On the Interaction of Representation Theory with Geometry and Combinatorics

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## Topics

Representation theory is one of the most vibrant fields of mathematics today. Starting with the pioneering work of Frobenius, Burnside and Schur, its history is a story rich in innovation and implementation of techniques from throughout mathematics. By its very nature, representation theory lies at the intersection of several fields: algebra, Lie theory, algebraic geometry, topology, number theory, differential geometry, combinatorics, harmonic analysis and mathematical physics.

The Hausdorff Trimester Programme provided a forum and meeting place for exchange of ideas and techniques, focussing on topics connecting representation theory with different areas of mathematics. It brought together leading experts and young researchers from representation theory, it enhanced research in representation theory by adding expertise from geometry and combinatorics and, at the same time, it strengthened the exchange between different research groups and networks.

## Goals

Great emphasis was put on achieving an interactive and inspiring working atmosphere, for a variety of participants working in representation theory, but with different background and at different stages in their mathematical career. After a long time of specialisation in this area, the main goal was to enhance the cooperation between the various specialised groups. The trimester at the HIM offered a perfect enviroment for achieving this goal. Leading researchers from different branches contributed to enhancing recent

developments in representation theory and to paving the way for new research directions. In a framework of workshop, winter school, lecture series and seminar, and in many informal meetings of small groups, the participants discussed projects and strategies and exchanged knowledge and experience. The special atmosphere of the institute very much supported this informal approach to forming collaborations, aiming at a synthesis of different concepts by interaction and cross-fertilisation. Some examples of collaboration and synergy will be given below.

The focus was on the following research topics, and on their interactions:

- Geometrisation, categorification and integral canonical structures
- Homological aspects of representation theory
- Representations of Cherednik algebras
- (Finite dimensional) algebras and cluster algebras
- Infinite-dimensional Lie algebras and related algebraic structures
- Combinatorics of flag varieties and Schubert calculus
- Hecke algebras, Koszul duality and tilting theory

## Organisation

The selection of the participants was quite a challenge, since the quality of applications was extremely high. The number of strong applications from postdocs and young researchers was more than twice the number to be accommodated in the programme. The high quality of the programme attracted in addition to the standard applications also a large number of requests for support for short term visits, which almost all had to be rejected.

The organisers reacted to the strong interest of young researchers by starting the programme with an introductory **winter school** addressing in particular PhD students. More than 80 participants came to the 5 lecture series given by international specialists in the crowded HIM seminar room. They covered some of the most influential developments in the area in the last years; 3 out of 5 speakers had been invited to an International Congress (ICM), two of them in 2010:

- Jonathan Brundan (USA): Finite W-algebras and primitive ideals
- Sergey Fomin (USA): Cluster algebras, total positivity, and triangulated surfaces
- Iain Gordon (UK): Rational Cherednik algebras
- Arun Ram (Australia): Combinatorics of flag varieties
- Claus Michael Ringel (Germany / China): The indecomposable representations of a finite quiver

During the trimester, three **lecture series** proved to be very attractive to large numbers of junior and senior participants. These lecture series (8 hours each) were running end of January to mid February, mid February to mid March and in the first half of April:

- Olivier Schiffmann (France): Hall algebras
- Alexander Kleshchev (USA): Representation theory of symmetric groups and related Hecke algebras
- Shrawan Kumar (USA): Geometry of Schubert varieties and Demazure character formula

One of the speakers had been invited to the 2010 ICM.

End of March, a large **international workshop** took place with almost 90 participants. Lectures were given by participants of the trimester, many of whom reported on progress made during the past few months at HIM, as well as by international experts just coming for the workshop. The topics of the lectures ranged from new concepts in representation theory (e.g. Kashiwara on higher category theory methods in representation theory) and far reaching programmes (e.g. Fiebig and Soergel on modular representation theory and generalised Koszul duality) over strong applications (e.g. Reineke on Gromow-Witten invariants and Euler characteristics of quiver moduli) and unexpected connections (e.g. Lam on electrical networks and Lie theory) to solutions of longstanding problems (e.g. Bongartz on the existence of indecomposable representations in all dimensions, for finite dimensional algebras of infinite representation type). For one postdoc in the trimester programme,

Qunhua Liu, it was her first plenary lecture at a major conference. She talked about lifting problems for recollements and the notion of simplicity of derived module categories.

The four months of the programme were structured by these highlights. Moreover, the regular seminar, running almost every Monday and Thursday, often with two talks in one session, offered more than 40 talks, which were very well-attended not only by participants of the programme, but also by visitors, staff and students from Bonn and neighbouring universities. In particular, many young participants established new contacts by presenting their results in the seminar. Also, new collaborations and interactions between different areas and research groups got started through such seminar talks. The HIM building proved to be an ideal place for collaborative research and the informal and inviting atmosphere very much facilitated new contacts, also between senior and junior researchers. In particular, there has been a surprisingly large number of discussions, exchange and even collaborations involving both senior and junior partipicants, but also students from Bonn, Köln and elsewhere.

## Examples of results

The programme created, often unexpected, collaborations and outcomes. For instance Kleshchev (USA), Mathas (Australia) and Ram (Australia) started a successful collaboration on universal Specht modules for cyclotomic Hecke algebras.

In the context of the representation theory of Hecke algebras, several new concepts and connections were developed and advanced during the programme. The notion of quiver flag varieties was one of the common themes of the programme. For instance, Kashiwara (Japan) developed the notion of Quiver Hecke superalgebras and their associated 2-categorifications and Brundan (USA) and Stroppel (Germany) produced graded versions of centraliser algebras. As an application they found tensor product decompositions for the general linear Lie superalgebra by connecting it with Deligne's universal monoidal category. Geiss (Mexico), Leclerc (France) and Schröer (Germany) determined cluster structures on quantum coordinate rings.

Hard conjectures around multiplicity formulae, decomposition numbers and related combinatorics of crystals and canonical bases could be proved. Gordon (UK) and Losev (USA) as well as Stroppel and Webster (USA) used crystal bases to prove conjectures on the structure of rational Cherednik

and q-Schur algebras. Morse and Schilling (USA) were able to derive a combinatorial formula for fusion coefficients. All these approaches rely on the combinatorics of certain Fock spaces and are connected with the geometry of quiver flag varieties.

As a direct consequence of a seminar talk by Evgeny Feigin (Russia) on degenerate flag varieties, Feigin, Reineke (Gemany) and Cerulli-Irelli (Italy) started a joint project to investigate the connection between degenerate flag varieties and quiver Grassmannians for type A quivers, leading to new combinatorial description of the large and small Schröder numbers and their q-analogues.

Major progress in tilting theory, resulting in an unexpected extension of Happel's theorem to infinitely generated tilting modules, was obtained by Chen and Xi (both China), stimulated by close exchange with Angeleri (Italy), Koenig (Germany), Liu and Yang (the two last-named being postdocs in the programme) as well as with Ringel. In the same line of research, Liu and Yang were able to prove that blocks of group algebras of finite groups are derived simple.

Chari (USA), Fourier (Germany), Kuz (Germany) and Manning (USA) have been working on the representation theory of (twisted) loop algebras and current algebras. In particular they established an injective map from the global Weyl modules for twisted loop algebras into a direct sum of global Weyl modules for untwisted loop algebras. They also found $\mathbb{Z}^\ell$-graded versions of Kirillov-Reshetikhin modules.

Henke (UK) and Koenig determined the endomorphism algebras of tensor powers of symmetric powers over symplectic and orthogonal groups, by putting a classical problem of invariant theory into the context of a new line of research on Brauer algebras and their Schur algebras.

Gaussent (France), Klostermann, Littelmann and Nguyen (Germany) worked on the connection between MV-cycles, the combinatorics of Hall-Littlewood polynomials and the geometry of Schubert varieties in the affine Grassmannian. Klostermann translated the formula for Hall-Littelwood polynomials found by Gaussent and Littelmann into the combinatorial language of tableaux and was able to show that this formula coincides in type A with the classical formula by Macdonald.

The standard PBW-filtration of an irreducible representation of a simple algebraic group leads to a degenerate flag variety. This variety can be viewed as flag variety for a deformed group which is a semidirect product of a commutative group and a large unipotent subgroup. Feigin (Russia),

Finkelberg (Russia), Fourier (Germany) and Littelmann (Germany) developed combinatorial and geometric tools to investigate the filtration as well as the geometric objects.

In the results described above known combinatorial numbers and their q-analogues get geometric or representation theoretic interpretations. Generalisations and interesting new combinatorial formulae were developed using geometry and representation theory.

The programme also produced mutual benefit with the current DFG priority programme (SPP 1388) in representation theory. Moreover, it strengthened the ties with leading international researchers. For instance, one senior participant, Alexander Kleshchev, obtained a Humboldt prize during the trimester. The new and exciting topic of Quiver Hecke algebras or KLR-algebras, which before the programme had not yet been taken up by the German research community, is in 2012 for the first time the topic of an international workshop in Germany (to be held in Stuttgart), and it is now also the topic of working groups and seminars at German universities. Further activities, such as a workshop at the interface of representation theory and topology, to be held in Bonn in 2013, will support the sustainability of the collaborations initiated during the trimester. Sustainability is also documented by the good number of invitations to young participants to visit international experts they had met during the trimester.

Many of the concepts developed at the trimester programme have now become standard topics at conferences and in research projects in representation theory. The programme has had significant impact on the field.