Research group final report
for the HIM Junior Trimester Program on Stochastics Fall 2010

Disordered systems and extreme value theory

July 29, 2011

The research group aimed at collaborative research on disordered systems and extreme values of random fields on discrete and continuous structures. The group was composed of Louis-Pierre Arguin, Onur Gün, Zakhar Kabluchko, Nicola Kistler, and Anton Klimovskyi. The specific research objectives were first to describe the statistics of extreme values of particular stochastic processes that are important in statistical physics, the so-called *disordered systems*, and in probability to a broader extent. Such models include for example spin glass models, branching Brownian motion, and Gaussian processes with logarithmic correlations. These random fields can be seen as a collection of highly-correlated random variables, the *energies*, indexed by the *states* of the system. A second objective was to understand the dynamics of these systems. This latter problem can be translated in terms of random walks in random environment. (Here the states of the walk are the states of the system and the random environment are the random energies of the states.)

The junior research program at HIM has been a key moment for progresses in the objectives stated above. A list of works that are the direct consequences of the discussions and work performed at the HIM are listed below. We summarize here the contributions. Anton Klimovskyi's research during his stay was mainly concentrated around the analysis of the extremes of high-dimensional Gaussian fields with isotropic increments, that is whose correlations present a rotational symmetry. He proved and analyzed a computable saddle-point variational representation for the free energy in the infinite-dimensional limit [6, 7]. In particular, he gave a rigorous proof of an important formula due to Fyodorov and Sommers for the free energy of a particle in a rotationally invariant box. Zakhar Kabluchko has analyzed the distribution of the maximum of a family of Gaussian fields [4], those with $1/f^{\alpha}$-noise, that appear in many applications. He also investigated the statistics of values close to the maximum of processes with hierarchical correlations such as branching Brownian motion [5]. Louis-Pierre Arguin and Nicola Kistler worked mainly on branching Brownian motion (BBM), a Gaussian process with a tree-like correlation structure. Correlations of BBM are high, of the order of the variance, and it was unknown until then how the correlations affect the distribution of values close to the maximum. The work done at HIM unveiled, somewhat surprisingly, a Poissonian statistics close to the maximum (Poissonian statistics are typical of independent variables not correlated ones) [1]. The work has later led to a complete description of the statistics in terms of Poisson cluster process [2]. Finally, Onur Gün has made some progress with Véronique Gayrard on the dynamics of GREM models, a particular kind of spin glasses that is expected to present a good approximation of more physical models such as the Sherrington-Kirkpatrick models.

The aforementioned results could not have been possible without the support of HIM and its staff. They are the consequences of the interactions during the activities organized at HIM. An important one was the workshop *Disordered systems and extreme value statistics* we organized in September 2010. Twelve researchers from all over the world participated. Prof. Yan Fyodorov from the University of Nottingham gave a mini-course on Gaussian processes with logarithmic correlations that has been very influential for the discussions conducted in the rest of the semester. Another important activity was a mini-course given by Onur Gün and Nicola Kistler on *the statistics and dynamics of disordered systems*. Finally, all the participants of our group, and one guest Prof. Véronique Gayrard, have presented at the weekly seminars of the HIM junior program.

References

[1] Arguin, Louis-Pierre, Bovier, Anton, & Kistler, Nicola. Poissonian Statistics in the Extremal Process of Branching Brownian Motion. *Submitted to Annals of Applied Probability, arXiv:1010.2376.*

[2] Arguin, Louis-Pierre, Bovier, Anton, & Kistler, Nicola. The extremal process of Branching Brownian Motion. *Submitted to Inventiones Mathematicae, arXiv:1103.2322.*

[3] Arguin, Louis-Pierre, & Damron, Michael. Short-range spin glasses and Random Overlap Structures. *J. Stat. Phys.* 143 (2011) 226-250.

[4] Kabluchko, Zakhar. The maximum of the Gaussian $1/f^{?}$-noise in the case ? < 1. *Preprint 2010, arxiv.org:1010.1902.*

[5] Kabluchko, Zakhar. Persistence of competing systems of branching random walks *Preprint 2011, arxiv.org:1103.5865.*

[6] Klimovsky, Anton. High-dimensional Gaussian processes with isotropic increments seen through spin glasses. *Preprint, 2011.*

[7] Klimovsky, Anton. Free energy of a particle in high-dimensional Gaussian potentials with isotropic increments: Fyodorov-Sommers formula *Preprint, 2011*

Probabilistic Aspects of Rough Paths: Research Report

Thomas Cass, Christian Litterer and Phillip Yam

Our project was centred around two related problems at the very core of the interplay between rough paths and stochastic processes. First we wished to develop a theory of rough differential equations on Lipschitz manifolds and explore applications to the study of diffusion processes on manifolds. The second problem concerned the integrability over pathspace of the Jacobian of the flow of random rough differential equations, a very well defined concrete problem, which frequently presents itself as an obstacle in the application of rough path techniques to stochastic analysis. We have fully achieved and in fact by now moved beyond the second objective and have made significant progress towards the first. Our work during the project has resulted in two papers [1] and [2] that are both available on the Arxiv.

In [1] we have (jointly with T. Lyons) developed an intrinsic coordinate free approach to rough paths on a manifold. The theory of rough paths is an extension of classical Newtonian calculus aimed at allowing models for the interaction of highly oscillatory and potentially non-differentiable systems. These systems take the form of differential equations driven by rough paths (RDEs) and naturally arise in non-linear geometric settings more general than Banach spaces. In [1] we regard rough paths as abstract objects that have integrals against sufficiently regular one forms taking their value in any Banach space. To specify the assumptions on the one form, we impose sufficient regularity on the Lipschitz manifold which we are working with. We can prove that the abstract non-linear functionals defined to be rough paths on the manifold have a notion of support that is a continuous paths on the manifold. Consequently we obtain a full characterisation of the rough paths on a Lip-γ manifold by identifying them with the pushforwards of finitely many classical rough paths from the coordinate charts. Finally, we are able to develop a notion of rough differential equations on the manifold and show the existence of solutions to such equations under suitable conditions on the vector fields.

Our entire construction is fully equivalent to the usual definition of rough paths if the manifold is a finite dimensional vector space and is the first work using rough paths in a geometric context.

In [2] we establish sharp estimates on the integrability of the Jacobian of the flow $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ of an RDE. To understand the difficulty of this problem, we note from [5] that the standard deterministic estimate on $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ gives

$$\left|J^{\mathbf{x}}_{t\leftarrow 0}(y_0)\right| \leq C \exp\left(C \left|\left|\mathbf{x}\right|\right|^p_{p\text{-var};[0,T]}\right). \tag{1}$$

But in the case where $\mathbf{X}$ is a Gaussian rough path and $p > 2$ (i.e. Brownian-type paths or rougher) the Fernique-type estimates of [4] give only that $||\mathbf{X}||_{p\text{-var};[0,T]}$ has a Gaussian tail, hence the right hand side of (1) is not integrable in general. Worse still, the work Oberhauser and Friz [3] shows that the inequality (1) can actually be saturated for a (deterministic) choice of differential equation and driving rough path. However, for random paths that have enough structure to them (in particular for Gaussian paths) only a set of small (or zero) measure comes close to equality in (1). What is therefore needed (and what we provide!) is to recast the deterministic estimate in a form that allows us to more strongly interrogate the underlying probabilistic structure

Our results in [2] allow us to deduce the existence of moments of all orders for $J^{\mathbf{x}}_{t\leftarrow 0}(y_0)$ for RDEs driven by a class of Gaussian processes (including, but not restricted to, fBm with Hurst index $H > 1/4$). In fact, our main estimate shows much more than simple moment estimates, namely that the logarithm of the Jacobian has a tail that decays faster than an exponential.

The results we obtain in [2] are relevant to a number of important problems. Firstly, they are a necessary ingredient if one wants to extend the work of [6] and [7] on the ergodicity of non-Markovian systems. Secondly, they are also an important ingredient in a Malliavin calculus proof on the smoothness of the density for RDEs driven by rough Gaussian noise in the elliptic setting. Furthermore, it allows one to achieve an analogue of Hörmander's Theorem on the smoothness of the density for Gaussian RDEs in conjunction with a suitable version of Norris's Lemma (see [10],[11]). In this context, we remark that Hu and Tindel [9] have recently obtained a Norris Lemma for fBm with $H > 1/3$ and proved smoothness-of-density results for a class of nilpotent RDEs. Hairer and Pillai [8] have also proved Hörmander-type theorems for a general class of RDEs; their results are predicated on the assumption that the Jacobian has finite moments of all order. Hence, one application of this paper is to use our tail estimate together with the results in [9] or [8] to conclude that for $t > 0$ the law of Y_t (the solution to the stochastic differential equation) will, under Hörmander's condition, have a smooth density w.r.t. Lebesgue measure on $\mathbb{R}^e$ for a rich classes of Gaussian processes X including fBm $H > 1/3$.

While we were in Bonn, we had a very successful visit by Samy Tindel from Nancy. And we intend to undertake further work which builds directly on the results we initiated during our stay at the institute.

We would like to thank Professor Kreck and his staff for providing us with such a stimulating and supportive working environment at the Hausdorff Institute. The progress we made during our stay has allowed us move beyond our original objectives and has spawned new collaborations. It is likely to have a lasting impact on the direction of our future research.

References

[1] T. Cass, C. Litterer and T. Lyons: Rough Paths on Manifolds, arXiv:1102.0998v1, to appear in New Trends in Stochastic Analysis and Related Topics: A Volume in Honour of Professor K D Elworthy, Eds. H. Zhao and A. Truman, World Scientific, 2011

[2] T. Cass, C. Litterer and T. Lyons : Integrability Estimates for Gaussian Rough Differential Equations, arXiv:1104.1813v4, preprint 2011

[3] Friz, P., Oberhauser, H.: Rough path limits of the Wong-Zakai type with a modified drift term. J. Funct. Anal. 256 (2009), no. 10, 3236–3256

[4] Friz, P., Oberhauser, H.: A generalized Fernique theorem and applications, Proceedings of the American Mathematical Society. 138, no. 10, 3679-3688, (2010)

[5] Friz, P., Victoir, N.: Multidimensional stochastic processes as rough paths. Theory and applications. Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010

[6] Hairer, M., Mattingly, J.C. : Ergodcity of the 2D Navier-Stokes Equations with Degenerate Stochastic Forcing, Annals of Mathematics, vol. 164, (2006), no. 3

[7] Hairer, M., Pillai N.S. : Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion, Ann. Inst. H. Poincaré Probab. Statist. Volume 47, Number 2 (2011), 601-628.

[8] Hairer, M., Pillai, N.: Regularity of Laws and Ergodicity of Hypoelliptic SDEs Driven by Rough Paths. ArXiv:1104.5218v1

[9] Hu Y., Tindel, S.: Smooth density for some nilpotent rough differential equations. ArXiv 1104.1972.

[10] Norris, J.: Simplified Malliavin calculus. Séminaire de Probabilités, XX, 1984/85, 101–130, Lecture Notes in Math., 1204, Springer, Berlin, 1986

[11] Nualart, D.: The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006

Final report

Carlo Marinelli

July 31, 2011

In the period September 1 to December 31, 2010 a group composed by Eulàlia Nualart, Lluís Quer-Sardanyons, Zeev Sobol and myself was hosted by the Hausdorff Research Institute for Mathematics (HIM), in the framework of a Junior Trimester Program on Stochastics.

During this period we have mostly worked on a problem of existence and regularity of the law for solutions to semilinear stochastic parabolic equations of the type

$$du(t,x) - \Delta u(t,x) + f(u(t,x)) = B\,dW(t,x), \qquad u(0,x) = u_0(x), \tag{1}$$

with $0 \leq t \leq T$, $x \in D$, where D is a bounded regular domain of $\mathbb{R}^d$, W stands for a space-time Wiener noise, $f : \mathbb{R} \to \mathbb{R}$ is an increasing function (not necessarily Lipschitz continuous) of polynomial growth, and B is a bounded linear operator on $L^2(D)$ such that the stochastic convolution

$$W_A(t) := \int_0^t S(t-s)B\,dW(s)$$

is well-defined for all $t \in [0,T]$. Here S stands for the semigroup in $L^2(D)$ generated by Δ, and W denotes the cylindrical Wiener process on $L^2(D)$ corresponding to the $d+1$-dimensional Brownian sheet in (1).

Assuming that the stochastic convolution is continuous in space and time and that $u_0 \in C^0(\overline{D})$, one has that (1) admits a unique mild solution, which is also (pathwise) continuous in space and time. Our main concern has been to investigate existence and regularity of the density of the real random variable $u(t,x)$ with respect to Lebesgue measure, for $(t,x) \in D_T :=]0,T] \times D$ arbitrary but fixed. We have used tools of the theory of (nonlinear) monotone operators and of Malliavin calculus.

To the best of our knowledge, all results related to this problem are limited to SPDE with very regular coefficients (typically f, $B \in C_b^1$). In fact, smoothness of coefficients ensures that one can write (at least formally) an equation for the Malliavin derivative $Mu(t,x)$ of $u(t,x)$ rather easily, and that one can derive from this equation enough estimates on $Mu(t,x)$ to infer that $u(t,x)$ admits a (regular) density. This scheme does not work, however, for equations with non-Lipschitz coefficients, essentially because the chain rule for the Malliavin derivative becomes inapplicable. The strategy we have adopted is to approximate the nonlinearity f in (1), thus obtaining the regularized equation

$$du_\lambda(t,x) - \Delta u_\lambda(t,x) + f_\lambda(u_\lambda(t,x)) = B\,dW(t,x),$$

where f_λ stands for the Yosida approximation of f, which is a Lipschitz continuous function converging pointwise to f as $\lambda \to 0$. One can now prove that $u_\lambda(t,x)$ is differentiable in the sense of Malliavin, and its derivative $Mu_\lambda(t,x)$ satisfies a suitable equation. We have been able to show, establishing a priori estimates for $Mu_\lambda(t,x)$ and using properties of the Malliavin derivative, to show $u(t,x)$ is Malliavin differentiable and that $Mu_\lambda(t,x)$ converges (weakly) to $Mu(t,x)$. Furthermore, passing to the limit in the equation satisfied by $Mu_\lambda(t,x)$, it has been possible to show that $Mu(t,x)$ satisfy an integral equation, from which the necessary estimates have been obtained to prove that $Mu(t,x)$ satisfy the sufficient conditions of the Bouleau-Hirsch criterion, thus concluding that $u(t,x)$ is absolutely continuous with respect to Lebesgue measure. An important ingredient in the above procedure is an equivalence result between mild and random field solutions to equation (1), which could also be of independent interest.

Further work is in progress in the following directions: (i) regularity of the density of $u(t,x)$ assuming more regularity on f (but without imposing any Lipschitz condition); (ii) extension of the results to the case of f being the sum of an increasing continuous function and a locally Lipschitz function of linear growth; (iii) multiplicative noise; (iv) more general second-order elliptic operators replacing the Laplacian.

Let me also briefly recall other research projects that have been carried out, at least in part, by members of my group during the stay at HIM:

1. E. Nualart and Ll. Quer-Sardanyons completed the work on a project they had already initiated aimed at establishing lower and upper Gaussian bounds for the probability density of the mild solution to the nonlinear stochastic heat equation in any space dimension, driven by a Gaussian noise which is white in time with some spatially homogeneous covariance. A corresponding paper is now under revision at *Stochastic Processes and their Applications.*

2. Z. Sobol has worked on a theory of SDE (on finite and, to some extent, infinite dimensional, state spaces) solvable with processes of finite lifetime. This includes a) a special topology on the space of paths with finite lifetime, which differs from the one used so far for Markov processes with finite life time; b) the corresponding theory of random processes, local martingales, and Ito processes; c) well-posedness of SDEs and martingale problems. As a corollary, he has obtained results on existence of a semigroup generated by the corresponding Kolmogorov operator and spatial Lipschitz continuity of the solution to the (backward) Kolmogorov equation.

3. Z. Sobol and I worked on the pricing of American options by an analytic approach. In particular, we characterized the value function of a general optimal stopping problem by means of a corresponding semilinear parabolic PDE with a monotone discontinuous absorption.

4. I continued my collaboration with A. Eberle (IAM, Bonn) on the rigorous study of the asymptotic properties of a class of MCMC algorithms. Part of the results will appear soon in *Probability Theory and Related Fields.*

Thanks to generous support by HIM, we organized two series of lectures, one held by Hatem Zaag (Paris XIII) on *Blow-up for the semilinear wave equation*, and a second one held by Thomas Duyckaerts (Cergy-Pontoise) on *Dynamics of the energy-critical focusing wave equation*. Moreover, the following seminar talks have been given: *Hitting probabilities and capacity for SPDEs*, by E. Nualart; *Stochastic integrals and SPDEs I & II*, by Ll. Quer-Sardanyons; *Tools for SPDEs I & II*, by myself.