# Report for HIM junior program on Analysis: Calculus of Variation and Image processing

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Starting from the well-known Mumford and Shah image segmentation model [3], and Total Variation minimizing (TV) functional [4],

$$\min \left\{ \int_\Omega |\nabla u| \, dx \quad + \quad \frac{\lambda}{2} \int_\Omega (u - u_o)^2 \, dx \right\} \tag{1}$$

variational model has been a crucial part of mathematical approaches to image processing. For instance, the Mumford-Shah [3] segmentation model,

$$\min \left\{ \alpha \mathcal{H}^1(\Gamma) + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \int_\Omega (u_o - K * u)^2 dx \right\} \tag{2}$$

is a perfect example showing the importance of the variational models in imaging research as well as the significance in the calculus of variation research.

For the Analysis program, we explored various analysis aspects of imaging problems, by bringing together international researches who are actively doing research on imaging problems and/or analysis and calculus of variations. The following main six people organized the program on "Calculus of Variation and Image Processing". The four members were analysts: Marco Barchiesi (Carnegie Mellon University, USA), Massimiliano Morini (Sissa, Italy), Luca Mugnai (Max Planck, Germany), Marcello Ponsiglione (Roma, Italy), and two members focused on mathematical image processing: Sung Ha Kang (Georgia Institute of Technology, USA) and Triet Le (Yale University, USA). Together we have explored various analytical questions on different imaging problems.

Through this program, we have invited various internationally renewed researchers and promoted interactions. The HIM program was very helpful and positive about these collaborations which tremendously helped the success of the program. We have organized the weekly seminar open to any researchers from the nearby universities. The list of visitors includes Francesco Maggi (University of Firenze, Italy), Ivano Primi (University of Heidelberg, Germany), Matthias Roegar (Max Planck, Germany), Giuseppe Riey (Italy), Riccardo March (Rome, Italy), Ha Quang Minh (Germany), Haomin Zhou (Georgia Tech, USA), Giovanni Bellettini (Italy), Antonin Chambolle

(France) and Simon Masnou (U of Pierre, France). The location of the HIM is also ideal for various collaborations and we had various interactions with local researchers from University of Bonn and different research groups.

Couple of research article resulted through this program [2] and [1]: **Colorization** refers to recovering color of gray scale images when only small regions with color are given. The term "colorization" was introduced by Wilson Markle who first processed the gray scale moon image from the Apollo mission. Through this collaboration, we proposed using Reproducing Kernel Hilbert space (RKHS) for image and video colorization. The setting of RKHS and its extensions are widely considered in machine learning, and this is the first work applying RKHS framework to image and video colorization, and the vectorial settings of RKHS are analyzed in [2]. The explicit solution is superior compared to any iterative methods, and the flexibility of choosing different kernels allows easy colorization on texture image as well as cartoon images. Colorization results show big improvements in speed and quality. With a close connection to applied analysis, in [1] we proposed a new model of image segmentation which captures the oscillatory boundary while denoising the image. A typical segmentation model uses the total length penalization as $\mathcal{H}^1(\Gamma)$ for regularization, and this term is replaced by the area of the boundary neighborhood in the new model. This is an initiating work on capturing the fine details of the boundary in image segmentation problem. By replacing the length term with an area of the $\epsilon$-neighborhood, this model allows the boundary with infinite perimeter, as long as it can be included in the $\epsilon$ bound. By computing the energy, it is shown that the model is able to denoise the image. Numerical experiments successfully capture the details of the oscillatory boundary while denoising homogeneous regions.

The HIM institute was truly ideal for mathematical collaborations and we were impressed with the support for research. The building: the office, many blackboards and many discussion/seminar rooms were quite convenient for mathematical discussions. The support for the visitors were essential for the success of the program, and all the staff member were very helpful. It was an impressive experience and we were all very grateful for all the HIM's support for research.

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# REPORT ON THE GROUP "SINGULARITIES IN MAGNETIC AND SUPERCONDUCTING MATERIALS" WITHIN THE HIM JUNIOR TRIMESTER PROGRAM "ANALYSIS", SEPTEMBER-DECEMBER 2008

**Participants.** R. Ignat (Orsay), M. Kurzke (Bonn), C. Melcher (Oxford), D. Spirn (Minnesota)

**Guests.** R. Moser (Bath, two weeks); E. Miot (Paris 6, one week)

**Scientific report.** The quantum mechanical theories for some effects in solid state physics can be effectively described by continuum variational principles. We discussed theories that describe superconductors or ferromagnets, mostly focusing on singularities such as walls or vortices and their dynamical behavior. Specifically, we focused on vortices, point defects in two-dimensional vector fields that carry a topological degree. As topological features, these vortices are quite stable, and it is interesting to see how the natural dynamical laws for various physical situations generate motion of vortices.

Our first main result was a motion law for a toy model, a "mixed flow" for Ginzburg-Landau vortices combining properties of a nonlinear Schrödinger equation and a gradient flow [1]. We did the main work on this during an extremely fruitful visit of R. Moser to the HIM that was the basis for a lot of further extensions. Our result states that the vortex motion can be described by an ODE as long as the initial configuration satisfies a certain well-preparedness condition. Independently from our work, E. Miot had derived the same ODE in a very similar setting, and it was very nice that it was extremely easy to invite her to HIM so we could discuss what happens when that well-preparedness condition is relaxed.

Combining the ideas used to derive the motion law discussed in [1] with other arguments from geometric analysis, we were able to treat a more difficult system the Landau-Lifshitz-Gilbert equation, a geometric PDE that lies between the Schrödinger map problem and the harmonic map heat flow, and where the main problem is to control "bubbling" of harmonic spheres. In the pleasant atmosphere of HIM, we discussed many approaches to the bubbling problem until we were able to rule it out under some special assumptions, and obtained a motion law [2]. This research is still ongoing, and we have improved results now, all based on work done at HIM.

For a more realistic model of superconductors, we found a motion law by coupling our previous results with some PDE estimates on the magnetic field [3]. The main parts of that article were finished during our time at HIM.

We were also able to continue and initiate further work on rigorous understanding of static properties of ferromagnets, working with other scientists in Bonn [5] and elsewhere [4]. Some of this work is still in progress: in [6] we attempt to rigorously treat a situation occurring in thin magnetic films using variational convergence techniques. Our estimates extend and combine much of previous work in this area.

At HIM, we benefitted very much from an open and relaxed working atmosphere. We also enjoyed the interaction with the simultaneously present image processing group, who ran a far more ambitious guest program. Many of the researchers in that group were interested in variational methods, leading to quite helpful discussions. Among other things, the collaboration [7] has its roots in the meeting at HIM.

## References

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# REPORT ON JUNIOR TRIMESTER PROGRAM "ANALYSIS", GROUP: TOPICS IN THE THEORY OF FUNCTIONAL EQUATIONS

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## 1. About the research plan

Our research plan contained the following four main topic:

**Functional equations on Abelian groups.** In [1] Boros and Daróczy examined the following so called composite type functional equation:

$$f(x+2f(y)) = f(x) + f(y) + y\,, \qquad (x, y \in G)\,,$$

where $G$ is an arbitrary Abelian group and $f : G \to G$. They prove that $f$ must be a homomorphism if there is no element of order two in $G$. They also formulated an open problem: solve the symmetric version of the above mentioned equation, namely find all solution of equation

$$f(x+2f(y)) + f(y+2f(x)) = 2f(x) + 2f(y) + x + y\,, \quad (x, y \in G)\,,$$

where $G$ is an arbitrary Abelian group and $f : G \to G$.

Firstly, we would like to prove that $f$ is a homomorphism also in the case, when $G$ has element(s) of order two. On the other hand we would like to give a more detailed description such homomorphisms.

Secondly, we would like to solve the symmetric version.

**Minkowski and reserved-Minkowski-type inequalities on homogeneous means, comparability of means.** In [2] Daróczy made the concept of the following class of homogeneous mean values:

$$\mathcal{D}_{\alpha,p}(x,y) := \begin{cases} \left(\frac{x^p+\alpha(\sqrt{xy})^p+y^p}{\alpha+2}\right)^{1/p} & \text{if } p \neq 0\,,\ -1 \leq \alpha < \infty \\ \mathcal{G}(x,y) := \sqrt{xy} & \text{if } p = 0 \text{ or } \alpha = \infty \end{cases} \qquad x, y \in \mathbb{R}_+.$$

It is a natural question to examine the comparability problem in this class, namely to find all parameters $\alpha\,, \beta\,, p\,, q$ such that $\mathcal{D}_{\alpha,p} \leq \mathcal{D}_{\beta,q}$ holds. Using this inequality and the ideas of Czinder and Páles (see [4] and [3]) we can examine Minkowski and reserved-Minkowski-type inequalities in this class. More precisely, we are looking for parameters (as far as possible the best parameters) $\alpha, \beta, \gamma, p, q, r$ such that the following inequalities hold

$$\begin{aligned} \mathcal{D}_{\alpha,p}(x_1+x_2, y_1+y_2) &\leq \mathcal{D}_{\alpha,p}(x_1,y_1)+\mathcal{D}_{\alpha,p}(x_2,y_2), \\ \mathcal{D}_{\alpha,p}(x_1+x_2, y_1+y_2) &\leq \mathcal{D}_{\beta,q}(x_1,y_1)+\mathcal{D}_{\gamma,r}(x_2,y_2) \end{aligned}$$

for all positive $x_1, x_2, y_1, y_2$.

**Stability of Hermite–Hadamard inequality.** The classical Hermite–Hadamard inequality, under some regularity assumptions, characterizes the convexity. That is the convex functions $f : D \to \mathbb{R}$ satisfy the so-called Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\big(tx+(1-t)y\big)dt \qquad (x, y \in D),$$

in other words, the integral average of the values of the function $f$ over a segment $[x, y]$ is nonsmaller than the value of the function at the midpoint of that segment. The converse is also known to be true i.e., if a continuous $f$ satisfies this inequality then it is also convex.

In the paper of Házy and Páles [HP08] the connection between the stability forms of the functional inequalities related to Jensen-convexity, convexity, and Hermite–Hadamard inequality when the stability term is not a constant but it depends on the closeness of the variables $x$ and $y$ was investigated. In other words the function $f : D \to \mathbb{R}$ satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta_J(\|x-y\|),$$

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\big(tx+(1-t)y\big)dt + \delta_H(\|x-y\|),$$

and

$$f\big(tx+(1-t)y\big) \leq tf(x)+(1-t)f(y)+\delta_C(t, \|x-y\|),$$

were considered, where $t \in [0,1]$, $x, y \in D$, $\delta_J, \delta_H : [0,\infty[ \to \mathbb{R}$, and $\delta_C : [0,1] \times [0,\infty[ \to \mathbb{R}$ are given functions called the stability terms. Our aim is to extend and generalize these results to the right-hand-side of Hermite–Hadamard inequality, that is to investigate the functions $f : D \to \mathbb{R}$ satisfying

$$\int_0^1 f\big(tx+(1-t)y\big)dt \leq \frac{f(x)+f(y)}{2} + \delta_H(\|x-y\|) \qquad (x, y \in D).$$

**Stability of generalized convexity.** One of the early results on regularity properties of convex functions is the so-called Bernstein–Doetsch theorem, which deduces convexity from Jensen-convexity and local upper boundedness property.

A natural extension of this theorem is to consider any kind of convexity combined with a weak regularity property and then to analyze the consequences. In this direction, where various generalizations of the Bernstein–Doetsch theorem were obtained by Nikodem and Ng [7], Házy and Páles in (see [8], [9], [10], [11], [12]).

The general aim is to obtain analogous statements for the solutions of various more general functional inequalities.

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## 2. Progress

There are two almost complete manuscripts in connection with the first and the second topic. We hadn't got time to deal with the third one. The last topic turned out to be the most successful. We wrote six publications regarding this theme:

- Pál Burai, Attila Házy and Tibor Juhász: *On approximately $s$-convex functions,* accepted for publication, Control and Cybernetics.
- Attila Házy, Bernstein-Doetsch-type results for h-convex functions, accepted for publication, Math. Ineq. and Appl..
- Pál Burai and Attila Házy: *On approximately $h$-convex functions,* Journal of Convex Analysis **18/2** (2011), 447–454.
- Pál Burai and Attila Házy: *On Orlicz-convex functions,* Proceedings of the Twelfth Symposium of Mathematics and its Applications, Editura Politechnica, Temesvár, (2010), 73–79.
- Pál Burai and Attila Házy: *Bernstein-Doetsch type results for generalized convex functions,* Proceedings of the Twelfth Symposium of Mathematics and its Applications, Editura Politechnica, Temesvár, (2010), 118–124.

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## 3. Personal comments

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