Bowman-Bradley type identities for symmetrised MZV's

MZV Days at HIM

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1. Background: finite MZV's and symmetrised MZV's

Finite MZV's defined by Hoffman, Zhao, and others as follows

$$\zeta_p(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \pmod{p} \,,$$

truncating before p in the denominator.

Zagier then considered all ζ_p simultaneously, modulo 'finite differences', to define:

$$\zeta^{\mathcal{A}}(k_1, \ldots, k_r) \coloneqq \big(\zeta_p(k_1, \ldots, k_r) \pmod{p}\big)_p \in \mathcal{A} \coloneqq \prod_p \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_p \mathbb{Z}/p\mathbb{Z}$$

Since $\mathbb{Q} \hookrightarrow \mathcal{A}$ diagonally, this is a $\mathbb{Q}$-algebra. These MZV's are defined for *all* $k_i \in \mathbb{Z}$, but get some mixing of weight

$$\zeta_p(-1,3) = \sum_{0<m<n<p} \frac{m}{n^3} = \sum_{0<n<p} \frac{1}{2} n(n-1) \frac{1}{n^3} = \frac{1}{2}\zeta_p(1) - \frac{1}{2}\zeta_p(2)$$

Imposing $k_i \geq 1$ fixes this, and allows us to define the space of weight k finite MZV's, write $Z_{\mathcal{A}_k}$. Experiments suggest

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} = \underbrace{d_{k-3}}_{\text{weight } k-3 \text{ usual MZV's}}$$

Comparing dimensions, via $d_{k-3} = d_k - d_{k-2}$ suggests maybe

$$Z_{\mathcal{A},k} \cong \underbrace{\mathcal{Z}}_{\text{usual MZV's}} / \pi^2 \mathcal{Z}$$

A suggestion for defining this isomorphism is via the symmetrised MZV's

$$\zeta^{S,\bullet}(k_1, \ldots, k_r) \coloneqq \sum_{i=0}^{r} (-1)^{k_{i+1} + \cdots + k_r} \zeta^\bullet(k_1, \ldots, k_i) \zeta^\bullet(k_r, \ldots, k_{i+1}) \,,$$

for $\bullet = ⧢, *$-regularisation.

Proposition 1.1 (Kaneko-Zagier)**.**

$$\zeta^{S,⧢} - \zeta^{S,*} \in \pi^2 \mathcal{Z} \,,$$

so $\zeta^S = \zeta^{S,\bullet} \pmod{\pi^2}$ *is well defined.*

Then conjectural isomorphism $\mathcal{Z}_{\mathcal{A}} \to \mathcal{Z}/\pi^2\mathcal{Z}$ is given via

$$\zeta_S(k_1, \ldots, k_r) \mapsto \zeta_{\mathcal{A}}(k_1, \ldots, k_r)$$

On the finite side, we have

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Theorem 1.2 (Bowman-Bradley type - [SW16])**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be odd integers, and $c_1, \ldots, c_m$ be even integers, all ≥ 1. Then*

$$\begin{aligned}&\sum_{(\sigma,\tau)\in S_n^2} \zeta^{\mathcal{A}}(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}\widetilde{\sqcup\!\sqcup}\{c_1\}\widetilde{\sqcup\!\sqcup}\cdots\widetilde{\sqcup\!\sqcup}\{c_m\})\\ &= \sum_{(\sigma,\tau)\in S_n^2}\sum_{\rho\in S_m} \zeta_{\mathcal{A}}(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}\widetilde{\sqcup\!\sqcup}\{c_{\rho(1)}, \ldots, c_{\rho(m)}\})\\ &= 0\end{aligned}$$

Remark 1.3. Here $\widetilde{\sqcup\!\sqcup}$ means shuffle of the MZV arguments (not the iterated integrals), i.e.

$$\{a_1, a_2, \ldots, a_p\}\widetilde{\sqcup\!\sqcup}\{b_1, b_2, \ldots, b_q\} = a_1(\{a_2, \ldots, a_p\}\widetilde{\sqcup\!\sqcup}\{b_1, b_2, \ldots, b_q\}) + b_1(\{a_1, a_2, \ldots, a_p\}\sqcup\!\sqcup\{b_2, \ldots, b_q\}) .$$

For example,

$$\begin{aligned}\zeta(\{2,2\}\sqcup\!\sqcup\{3,5\}) &= \zeta(2,2,3,5) + \zeta(2,3,2,5) + \zeta(2,3,5,2)\\ &\quad + \zeta(3,2,2,5) + \zeta(3,2,5,2) + \zeta(3,5,2,2)\end{aligned}$$

Goal: corresponding result for symMZV's.

Remark 1.4. Some results already by Muneta (Kyushu MZV seminar)

$$\zeta^S(\{1,3\}^n\widetilde{\sqcup\!\sqcup}\{2\}^m) = \binom{m+n}{n}\frac{(-1)^n 2^{2m+2n+1}}{(2m+4n+2)!}\pi^{2m+4n} \equiv 0 \pmod{\pi^2}$$

(Here $\sqcup\!\sqcup$, $*$-regularisation are equal because there is no consecutive $1,1$ in the result.)
Murahara also has some unwritten results.

2. A general Bowman-Bradley 'type' identity

Theorem 2.1. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be odd integers, and $c_1, \ldots, c_m$ be even integers, all ≥ 1. Then*

$$\begin{aligned}&\sum_{(\sigma,\tau)\in S_n^2} \zeta^{S}(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}\widetilde{\sqcup\!\sqcup}\{c_1\}\widetilde{\sqcup\!\sqcup}\cdots\widetilde{\sqcup\!\sqcup}\{c_m\})\\ &= \sum_{(\sigma,\tau)\in S_n^2}\sum_{\rho\in S_m} \zeta^{S}(\{a_{\sigma(1)}, b_{\tau(1)}, \ldots, a_{\sigma(n)}, b_{\tau(n)}\}\widetilde{\sqcup\!\sqcup}\{c_{\rho(1)}, \ldots, c_{\rho(m)}\})\\ &= \sum_{(\sigma,\tau)\in S_n}\sum_{\substack{B=(B_1,\ldots,B_k)\\ B\in\Pi_{\leq 2}(m)}} (-1)^n 2^{\#B}\zeta(\{a_{\sigma(1)}+b_{\tau(1)}, \ldots, a_{\sigma(n)}+b_{\tau(n)}\}\widetilde{\sqcup\!\sqcup}\{c_{B_1}\}\widetilde{\sqcup\!\sqcup}\cdots\widetilde{\sqcup\!\sqcup}\{c_{B_k}\}) .\end{aligned}$$

Here we employ the following notation

$$\begin{aligned}&\Pi_{\leq 2}(m) := \{\text{ all partitions } (B_1, \ldots, B_k) \text{ of } \{1, \ldots, m\} \text{ with } \#B_i \leq 2 \} \text{ , and}\\ &c_{B_j} := \sum_{k\in B_j} c_k .\end{aligned}$$

Corollary 2.2. *The result vanishes modulo π^2, which matches the expectation under the $\zeta^{\mathcal{A}} \leftrightarrow \zeta^S$ correspondence.*

Proof. For arbitrary odd a_i, b_i, even c_i, we see the result is $0 \pmod{\pi^2}$: After summing over $(\sigma,\tau) \in S_n$, the ζ is symmetric in all arguments. Hence by the symmetric sum formula, we can write the result as a polynomial in

$$\zeta(\alpha_{\sigma,\tau,i,j}(a_{\sigma(i)} + \beta_{\tau(j)}) + \beta_k c_{B_k})$$

Since $a_i + b_j$ and $\sum_l c_l$ are all even, the result is an even zeta, which vanishes modulo π^2. $\square$

Sketch of Theorem. Purely combinatorial, and by induction. Very similar to the finite case. Case $m = 0$ corresponds to the following stuffle-algebra identity

$$\sum_{(\sigma,\tau)\in S_n^2} \left\{ \sum_{i=0}^n z_{a_{\sigma(1)}} \cdots z_{b_{\tau(i)}} * z_{b_{\tau(n)}} \cdots z_{a_{\sigma(i+1)}} - \sum_{i=1}^n z_{a_{\sigma(1)}} \cdots z_{a_{\sigma(i)}} * z_{b_{\tau(n)}} \cdots z_{b_{\tau(i)}} \right\}$$
$$= \sum_{(\sigma,\tau)\in S_n^2} (-1)^n z_{a_{\sigma(1)}+b_{\tau(1)}} \cdots z_{a_{\sigma(n)}+b_{\tau(n)}}$$

which is also proven by induction.

Then use the stuffle-product result

$$\zeta^S(\mathbb{k})\zeta^S(\mathbb{l}) = \zeta^S(\mathbb{k} * \mathbb{l}) \,,$$

and relate this to $\widetilde{ш}$ as follows

$$\zeta^S(\mathbb{k}\widetilde{ш}\{c\}) = \zeta^S(\mathbb{k} * c) - \sum_i \zeta(k_1, \ldots, k_i + c, \ldots, k_n) \,.$$

This allows us to shuffle in a single c at a time, to obtain the result. One obtains two level m versions with variables either $a_i/b_i + c_{m+1}$, or $a_i/b_i, c_j + c_{m+1}$. ☐

3. Corollaries and evaluations

From this can set $a_i = a$, $b_i = b$, to obtain

Corollary 3.1 (Bowman-Bradley)**.**

$$\zeta^S(\{a,b\}^n \widetilde{ш} \{c\}^m) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^n 2^{m-2i} \zeta(\{a+b\}^n \widetilde{ш} \{c\}^{m-2i} \widetilde{ш} \{2c\}^i) = 0 \pmod{\pi^2} \,.$$

When $m = 0$, we obtain

$$\zeta^S(\{a,b\}^n) = (-1)^n \zeta(\{a+b\}^n) \,,$$

which can be explicitly evaluated in each case using generating series results about $\zeta(\{even\}^n)$.

To go to higher m, we need to evaluate combinations like $\zeta(\{p\}^k \widetilde{ш} \{q\}^l \widetilde{ш} \{r\}^m)$. I'm not aware of any such results so far, but I can conjecture the following

Observation 3.2. For any $a, b, c \in \mathbb{Z}_{\geq 0}$, the following evaluation appears to hold

$$\zeta(\{2\}^a \widetilde{ш} \{4\}^b \widetilde{ш} \{6\}^c) = \frac{2^{1+2b+6c}(b+2c)!(1+a+2b+4c)!}{(1+2c)!a!b!(1+2b+4c)!(2+2a+4b+6c)!} \pi^{2a+4b+6c}$$
$$= \frac{2^{1+2b+6c}\pi^{2a+4b+6c}}{(1+2c)(2+2a+4b+6c)!} \binom{b+2c}{2c} \binom{1+a+2b+4c}{a}$$

Corollary 3.3. *(Assuming the above is accurate), the following evaluations hold*

(Muneta)
$$\zeta^S(\{1,3\}^n \widetilde{ш} \{2\}^m) = \binom{m+n}{n} \frac{(-1)^n 2^{2m+2n+1}}{(2m+4n+2)!} \pi^{2m+4n}$$
$$\zeta^S(\{3,3\}^n \widetilde{ш} \{2\}^m) = \frac{1}{2n+1} \binom{2n+m}{m} \frac{(-1)^n 2^{2m+6n+1}}{(2m+6n+2)!} \pi^{2m+6n}$$

Proof. The resulting binomial sums can be evaluated using the WZ-method. ☐

Remark 3.4. Not sure if there is a nice generating series proof of the above observation; the naïve generating series obtained by generalising the $\zeta(\{a\}^n)$ evaluation gives $\zeta(\{a\}^n)\zeta(\{b\}^l)$ type results instead.

Using the symmetric sum theorem, I can recursively reduce a proof of the above to proving the following Bernoulli identities, neither of which seems particularly easy to prove.

$$\sum_{n_a=0}^{a}\sum_{n_b=0}^{b}(-1)^{n_b}2^{2n_a+2n_b}B_{4+2n_a+4n_b}\binom{6+2a+4b}{4+2n_a+4n_b}\binom{1+a+2b-n_a-2n_b}{a-n_a}\binom{n_a+n_b}{n_a}$$
$$=\frac{-(b+1)}{2}\binom{3+a+2b}{a}$$

$$\sum_{n_a=0}^{a}\sum_{n_b=0}^{b}\sum_{n_c=0}^{c}\frac{(-1)^{n_b}2^{2n_a+2n_b}}{1+2c-2n_c}B_{6+2n_a+4n_b+6n_c}\binom{8+2a+4b+6c}{6+2n_a+4n_b+6n_c}$$
$$\binom{1+a+2b+4c-n_a-2n_b-4n_c}{a-n_a}\binom{b+2c-n_b-2n_c}{2c-2n_c}\underbrace{\binom{n_a+n_b+n_c}{n_a,n_b,n_c}}_{\text{multinomial}}$$
$$=\frac{2+2c}{3+2c}\binom{2+b+2c}{2+2c}\binom{5+a+2b+4c}{a}$$

(Murahara recently suggested a different recursion, and reduces this to a certain binomial sum identity. Hopefully this is more accessible.)

Beyond $\zeta(2\widetilde{\sqcup\!\sqcup}4\widetilde{\sqcup\!\sqcup}6)$, one necessarily encounters $\zeta(8)$, and like the evaluation of $\zeta(\{8\}^n)$, these evaluations become more difficult to find and write.

Theorem 3.5. *For $n \geq 0$, $R_\pm = 64(17 \pm 12\sqrt{2}) = 4^3(1 \pm \sqrt{2})^4$, and $\sigma\colon \sqrt{2} \mapsto -\sqrt{2}$ the Galois automorphism of $\mathbb{Q}(\sqrt{2})$, we have*

$$\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^0) = \tfrac{\pi^{\text{wt}}}{(\text{wt}+4)!}\Big\{R_+^n\Big((12+8\sqrt{2})\Big)\Big\}^{\sigma\leftarrow\text{Galois symmetrisation}}$$
$$\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^1) = \tfrac{\pi^{\text{wt}}}{(\text{wt}+4)!}\Big\{R_+^n\Big((60+42\sqrt{2})+n(80+56\sqrt{2})\Big)\Big\}^{\sigma}$$
$$\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^2) = \tfrac{\pi^{\text{wt}}}{(\text{wt}+4)!}\Big\{R_+^n\Big((168+118\sqrt{2})+n(440+310\sqrt{2})+n^2(272+192\sqrt{2})\Big)\Big\}^{\sigma}$$

Proof. Proven using $\zeta(\{8\}^n)$ as the base case, and summing up the Bernoulli sums using the generating series of Bernoulli polynomials. □

Observation 3.6. One finds that $\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^m)$, m fixed, appears to satisfy a linear recurrence relation of order $2m+2$, whose characteristic equation factors as

$$(\lambda - R_+)^{m+1}(\lambda - R_-)^{m+1} = 0\,.$$

So by finding the first $2m+2$ instances, one obtains further candidate results like

$$\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^3) = \tfrac{\pi^{\text{wt}}}{(\text{wt}+4)!}\Big\{R_+^n\Big((360+\tfrac{1015}{4}\sqrt{2})+n(\tfrac{3994}{3}+\tfrac{2819}{3}\sqrt{2})+n^2(1608+1136\sqrt{2})+n^3(\tfrac{1856}{3}+\tfrac{1312}{3}\sqrt{2})\Big)\Big\}^{\sigma},$$

and a general form

$$\zeta(\{8\}^n\widetilde{\sqcup\!\sqcup}\{2\}^m) = \tfrac{\pi^{\text{wt}}}{(\text{wt}+4)!}\Big\{R_+^n\sum_{j=0}^{m}\alpha_j n^j\Big\}^{\sigma},$$

some $\alpha_j \in \mathbb{Q}(\sqrt{2})$.

Unfortunately, not clear what the pattern is coefficients is. Moreover, some coefficients have large prime factors dividing their norm:

$$N_{\mathbb{Q}(\sqrt{2})}(\tfrac{3994}{3}+\tfrac{2819}{3}\sqrt{2}) = 2^1\cdot 3^{-2}\cdot 17\cdot 1721\,.$$

3.1. Miscellaneous results

It doesn't yet appear as if any analogue of cyclic insertion holds in general for symMZV's. Numerically, I have checked how Bowman-Bradley for $\zeta(\{1,3\} \sqcup\!\sqcup \{2\})$ decomposes into $\pi^\bullet$-pieces, but it doesn't seem so well structured yet.

However

Observation 3.7. The following evaluation

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \zeta^S(\{2\}^{a_{\sigma(1)}}, 3, \{2\}^{a_{\sigma(2)}}, 3, \ldots, 3, \{2\}^{a_{\sigma(2n+1)}}) = 2^{\mathrm{wt}+1} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+2)!}$$

appears to hold.

So there might be something interesting here...

The proof of 'generalised Bowman-Bradley' for ζ^S should give directly a different generalisation when c_i are arbitrary, and $n = 0$. Namely

$$\sum \zeta^S(\{c_1\} \widetilde{\sqcup\!\sqcup} \cdots \widetilde{\sqcup\!\sqcup} \{c_m\}) = \sum_{\substack{(B=B_1,\ldots,B_k) \\ B \in \Pi_{\leq 2}(m)}} \prod_j (1 + (-1)^{c_{B_j}}) \cdot \zeta(\{c_{B_1}\} \widetilde{\sqcup\!\sqcup} \cdots \widetilde{\sqcup\!\sqcup} \{c_{B_k}\}) \,,$$

where $c_{B_j} = \sum_{i \imath B_j} c_i$. Note, in particular, that the if any c_{B_j} is odd, the term vanishes. So one could write this as a sum over all partitions $B \in \Pi_{\leq 2}(m)$ such that every $c_{B_i} = 0 \pmod 2$.

Moreover, one can probably give a common generalisation (naturally with a more complicated expression), of these two results, to arbitrary a, b, c.

Nevertheless, one can give results like

$$\begin{aligned}
&\zeta^S(\{1\}^{\widetilde{\sqcup\!\sqcup} 2} \widetilde{\sqcup\!\sqcup} \{3\}^{\widetilde{\sqcup\!\sqcup} 2n}) \\
&= 2^{n+1}(2n-1)!!n!\big(\zeta(\{2\} \widetilde{\sqcup\!\sqcup} \{6\}^n) + 4\zeta(\{4,4\} \widetilde{\sqcup\!\sqcup} \{6\}^{n-1})\big) \\
&= 2^{2+7n}(2+n)\frac{n!(-1+2n)!!}{(4+6n)!}\pi^{2+6n} \\
&\equiv 0 \pmod{\pi^2}
\end{aligned}$$

$$\begin{aligned}
&\zeta^S(\{3\}^{\widetilde{\sqcup\!\sqcup} 2} \widetilde{\sqcup\!\sqcup} \{5\}^{\widetilde{\sqcup\!\sqcup} 2n}) \\
&= 2^{n+1}(2n-1)!!n!\big(\zeta(\{6\} \widetilde{\sqcup\!\sqcup} \{10\}^n) + 4\zeta(\{8,8\} \widetilde{\sqcup\!\sqcup} \{10\}^{n-1})\big) \equiv 0 \pmod{\pi^2}
\end{aligned}$$

References

[SW16] Shingo Saito and Wakabayashi. "Bowman-Bradley type theorem for finite multiple zeta values". In: *Tohoku Mathematical Journal* 68.2 (2016), pp. 241–251.