YOUNG RESEARCHERS NETWORK: HIGHER THETA FUNCTIONS

1. Overview

Theta functions are special automorphic forms attached to Howe dual pairs (H, G) of reductive groups. This workshop will be on the theme of *higher theta functions*, as introduced in the recent works [FYZ21a, FYZ21b], which concern an arithmetic geometry incarnation of theta functions in characteristic p. They are analogous to Kudla's *arithmetic theta functions*, but with the additional feature that there are "higher legged" versions.

Let us briefly summarize the (expected) fruits of this theory. First we begin with some classical motivation.

(1) (Modularity) The classical theta function is modular, meaning it is an automorphic form for $H \times G$.
(2) (Siegel-Weil) The classical *Siegel-Weil formula* relates the integral of a theta function over G to the special value of a Siegel-Eisenstein series for H.
(3) (Rallis inner product formula) The *Rallis inner product formula* relates the Petersson inner product of a theta function to the special value of an L-function.

The arithmetic theta function of Kudla is conjectured to enjoy suitable generalizations of these statements.

(1) (Modularity) The *arithmetic theta function*, initially defined as a formal q-series, should define an automorphic function of H valued in the Chow group of a Shimura variety attached to G, which we will denote Sh_G.
(2) (Arithmetic Siegel-Weil) The *arithmetic Siegel-Weil* formula should relate the integral of the arithmetic theta function over Sh_G to a special derivative of a Siegel-Eisenstein series for H.
(3) (Arithmetic inner product formula) The *arithmetic inner product* formula should relate the Beilinson-Bloch height self-pairing of an arithmetic theta function to the special derivative of an L-function.

Many of these statements are now proven in partial generality, thanks to works such as [BHK$^+$20] (for modularity), [GS19, LZ22a, LZ22b] (for the arithmetic Siegel-Weil formula), [LL21, LL22] (for the arithmetic inner product formula).

In the characteristic p story, there are higher theta functions Θ_r indexed by $r = 0, 1, 2, \ldots$. We expect each of these to enjoy

(1) A "higher modularity" property.
(2) A "higher arithmetic Siegel-Weil" formula.
(3) A "higher arithmetic inner product" formula.

For $r = 0$, Θ_r is a classical theta function and relates to special values. For $r = 1$, Θ_r is an arithmetic theta function and should relate to special derivatives. In general, Θ_r should relate to rth derivatives; thus the higher theta functions give access to the entire Taylor expansion of the analytic side.

Partial progress towards points (1) and (2) are known. In [FYZ21a], a higher Siegel-Weil formula is proved for the non-singular Fourier coefficients. In [FYZ21b], the higher theta functions are fully constructed (the novelty is the definition of the singular coefficients), enabling a conjectural formulation of the modularity property. Moreover, [FYZ21b] introduced the idea that special cycles should really be understood in terms of derived algebraic geometry. In [FYZ] the modularity is proved in the cohomology of a certain open subset. In addition to covering higher derivatives, the arguments from these papers are quite new, and have almost nothing to do with existing proofs in the Kudla program. The workshop will go through some of the ideas from these papers. Morning talks will be on material from [FYZ21a] and [FYZ21b]. Some afternoon sessions may feature bonus talks on [FYZ].

2. Schedule of talks

We encourage speakers to focus on simplifying special cases and examples that illustrate the main ideas.

2.1. **Day 1: classical motivation.** The first day will focus on classical constructions that motivate our work.

Talk 1: Theta functions. Please read [Liara, §1,2] and [Liarb, §2]. Define theta functions in terms of q-series and formulate the Siegel-Weil formula, with some concrete examples. Emphasize the special case where the Schwartz function φ is the indicator function of the integer lattice. (Overall, try to give examples rather than super general group-theoretic definitions.) Sketch the proof of modularity via Poisson summation, at least in a special case.

Talk 2: Arithmetic theta functions. Please read [Liara, §3, §4] and [Liarb, §4]. and Define special cycles on unitary Shimura varieties, following Kudla-Rapoport [KR14]. The technical details won't actually be so relevant for later talks, so please emphasize the big picture instead. Please use "a" instead of "T" for the Fourier parameter indexing special cycles. Explain why it is necessary to define special cycle classes in higher codimension (for non-singular terms) as "derived intersections" (and explain what this means). Formulate the Arithmetic Siege-Weil formula and the modularity conjecture.

2.2. **Day 2: shtukas, special cycles, and higher theta functions.** The second day will define moduli of Hermitian shtukas and the special cycles on them, and formulate the Modularity Conjecture for higher theta functions.

Talk 3: Hermitian shtukas and special cycles. Please read [FYZ21a, §6, 7]. Define the notions of Hermitian bundles, Hecke stacks, and Hermitian shtukas and their moduli spaces. State their geometric properties. Then define the special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$, explain their indexing by Fourier coefficients, state the basic geometric properties, and define the fundamental classes in the special case of [FYZ21a, §7.8], where $\mathcal{E}$ is a sum of line bundles and each a_{ii} is non-zero. Emphasize the analogy to Kudla-Rapoport cycles. Mention why the general case is subtler (than what happens for Kudla-Rapoport) and sketch briefly what needs to be done in that case to define special 0-cycles.

Talk 4: The modularity conjecture. (Requires familiarity with the material of Talk 3.) Please read [FYZ21a, §8] and [FYZ21b, §4] (skimming the parts of [FYZ21b, §2,3] that are necessary, but ignore the generalization to arbitrary similitude bundles). Define the Hitchin space $\mathcal{M}$ from [FYZ21a, §8]. Explain the $\mathrm{Sht}^r_{\mathcal{M}}$ construction and the relationship with $\mathcal{Z}^r_{\mathcal{E}}$, and use this to define the special cycles in general for non-singular coefficients. (Caution: there may be a small mistake/typo in [FYZ21a] about this, so look at the most recent version of [FYZ21b].) Using the Hitchin space, sketch the definition of cycles in the more general case of singular coefficients from [FYZ21b, §4], treating the case $a = 0$ in detail. Explain how to assemble these coefficients into a Fourier series following [FYZ2,

§4.5] and state the Modularity Conjecture. Explain what this has to do with automorphic forms, using Weil's uniformization.

Afternoon bonus talk: modularity in the case of 0 legs following [FYZ, §2].

2.3. **Day 3: derived special cycles.** The third day will sketch the derived algebraic geometry approach to special cycles.

Talk 5: Introduction to derived geometry. Please read [FYZ21b, §5.1, 5.2, 6.1] and [FH23, §2-4, Appendix A]. Introduce the basic ideas of derived algebraic geometry: simplicial commutative rings, cotangent complex, quasi-smoothness [FH23] and virtual fundamental classes for quasi-smooth morphisms [Kha19]. Emphasize the analogy between derived structure and non-reduced structure; target the talk towards an audience that has no prior experience

Talk 6: Derived special cycles. (Requires familiarity with the material of Talks 4, 5) Please read [FYZ21b, §5,6,7]. Sketch the construction of the fundamental classes for special cycles in terms of derived algebraic geometry. Try to give a sample calculation or two for concreteness. State the key result Theorem 6.5. You probably will not have time to give the proof, but try to give a heuristic argument of the most degenerate and least degenerate cases, using the Excess Intersection formula for the former. Illustrate how the derived theory is used to make the Octahedron Lemma from [YZ17, Theorem A.10] trivial, and to prove the Linear Invariance [FYZ21b, Theorem 7.1]. You may have to be a little informal in your presentation to convey the main ideas.

Afternoon bonus talk: Derived Fourier analysis following [FYZ, §5].

2.4. **Day 4: higher Siegel-Weil (analytic side).** The fourth day will commence the proof of the higher Siegel-Weil formula, starting with the analytic side.

Talk 7: Perverse sheaves and Springer theory Please read [Yun17, §1] and [FYZ21a, §3,4] (you may skip the involved computations). Give a brief and gentle introduction to perverse sheaves, saying something about: intersection complexes, intermediate extensions, small maps. Illustrate with Springer theory for $\mathfrak{gl}_n$, defining the Grothendieck alteration and the Springer action. (Possible references for this material are [HTT08, §8] and [CG10, §5].) Define the Springer sheaves associated to irreducible representations of W. Define the stacks $\mathrm{Coh}_d(X)$ and Herm_d and explain their local geometry.

Talk 8: Geometrization of local densities. (Requires familiarity with the material of Talk 8) Please read [FYZ21a, §2, §5]. State the connection between non-singular Fourier coefficients of Eisenstein series and local density polynomials following [FYZ21a, §2]. Explain the function-sheaf dictionary (a possible reference is [FKM14, §2,3]). Sketch the definition of local density sheaves from [FYZ21a, §5] and explain [FYZ21a, Theorem 12.2], perhaps saying a bit about the proof if there is time. You are encouraged to suppress normalization constants in order to convey the main ideas.

Afternoon bonus talk: the sheaf-cycle correspondence following [FYZ, §3,4]

2.5. **Day 5: higher Siegel-Weil (geometric side).** The fifth day will focus on the geometric side of the higher Siegel-Weil Theorem.

Talk 9: Hitchin fibration and intersection numbers (Requires familiarity with Talk 3) Please read [FYZ21a, §8, §11] and [YZ17, Appendix A]. Explain the Hitchin fibration and how it is used to calculate the degree of special cycles (with non-singular coefficients), which is [FYZ21a, Corollary 11.10]. Along the way, you will need to reconcile two different constructions of the special zero-cycles, one via the Kudla-Rapoport construction and one via Hitchin spaces. This is done via an elaborate argument in [FYZ21a, §10] using the Octahedron Lemma in [YZ17]; instead give a very short argument using the derived perspective (it is an easy instance Linear Invariance from Talk 6).

Talk 10: Comparison (Requires familiarity with Talk 7, 8, 9) Please read [FYZ21a, §8, §11, §12]. For the Hitchin fibration $f\colon \mathcal{M}_d \to \mathcal{A}_d$, explain the calculation of $f_! \mathbf{Q}_\ell$ following [FYZ21a, §8.4 and §11.2]. Then explain the Hecke action on it, and its computation in [FYZ21a, Proposition 11.7] and [YZ17, Proposition 8.3]. Summarize the situation following [FYZ21a, §12.1], and explain how the comparison is completed. This will involve stating [FYZ21a, Proposition 12.3], but you can omit the proof of [FYZ21a, Lemma 12.4].

Afternoon bonus talk: modularity in the cohomology of the generic fiber.

3. Advice for your talk

A compilation of useful advice for similar workshops can be found at `https://math.mit.edu/events/talbot/talk-advice.pdf`. To this I would add the following: I find that talks have a higher tendency to be successful if the speaker does *not* rely on notes when delivering the talk. (You can keep your notes nearby to check in case you forget something, but I suggest not copying off of them.) This usually forces the speaker to craft the talk into a coherent story, and prevents the speaker from writing down overcomplicated formulas that the audience will not be able to absorb.

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