BONN LECTURES

MATTHEW EMERTON AND TOBY GEE

1. Lecture schedule

This schedule is provisional, and the precise order of lectures and division of material may change as the week goes on.

Lecture 1 (ME)

- A sketch of the $\ell \neq p$ picture.
- The rank 1 case, [EG19a, §7].
- Fontaine–Laffaille stacks for $\mathrm{GL}_2(\mathbf{Q}_p)$, and more generally.
- A sketch of the main statements of [EG19a].

Lecture 2 (TG) Definitions related to étale φ-modules and étale (φ, Γ)-modules. [EG19a, §2].

- The coefficient rings and their various relationships.
- Definitions of étale φ-modules, (φ, Γ)-modules, Breuil–Kisin–Fargues modules etc.
- Almost Galois descent.

Lecture 3 (ME) Stack-theoretic material related to the basic definitions and constructions of our various stacks [EG19b, §2 & 3].

- Basic definitions of categories fibred in groupoids (emphasizing the intuition of a "presheaf of groupoids"), stacks in groupoids (emphasize the intution of a "sheaf of groupoids"), morphisms, algebraic stacks, Ind-algebraic stacks, formal algebraic stacks, etc.
- Versality and versal rings.
- Scheme-theoretic images, in the context of morphisms of algebraic stacks, and then in the more general context of [EG19b].
- Effectivity statement for the universal étale φ-module over a finite height deformation ring [EG19b, §5].
- Criterion for Noetherianness, [Eme, §11].

Lecture 4, 5 and 6 (TG) [EG19b, §5] and [EG19a, §3]: definitions of the stacks themselves (excluding the crystalline/semistable stacks), and basic properties building on the work of Pappas–Rapoport [PR09] (which in turn uses the results of [Dri06] on *fpqc* locality of Tate modules).

This will in particular combine the stack-theoretic and φ-module theoretic material introduced in the preceding two lectures, and explain the role of Galois deformation rings in proving the Ind-algebraicity of the stacks.

Crystalline and semistable stacks [EG19a, §4]:

- Relating (φ, Γ)-modules to Breuil–Kisin modules, via almost Galois descent.
- Characterization of semistability via Breuil–Kisin–Fargues modules.
- Canonical Galois actions on Breuil–Kisin modules.

- Definitions and properites of crystalline and semistable stacks.
- Inertial types and Hodge–Tate weights., etc.

Lecture 7 (ME) Families of étale (φ, Γ)-modules and their applications [EG19a, §5].

- Local Galois cohomology background, and then the general theory of the Herr complex.
- Families of extensions.
- Formality of $\mathcal{X}$.
- Obstruction theory and Noetherianness of $\mathcal{X}$.

Lecture 8 (ME) Crystalline lifts and the structure of $\mathcal{X}_{\text{red}}$ [EG19a, §6]..

- Motivational discussion of the two-dimensional $G_{\mathbf{Q}_p}$ très ramifiée case.
- Geometric lemmas.
- Crystalline lifts.

Lecture 9 (TG) Application of crystalline lifts to the structure of $\mathcal{X}_{\text{red}}$, [EG19a, Thm. 6.5.1]. The weight part of Serre's conjecture and the geometric Breuil–Mézard conjecture, concentrating on the two-dimensional case [EG19a, §8].

Lecture 10 (ME) Associated moduli spaces and the Bernstein centre:

- (Non)-general theory of moduli spaces for Artin stacks.
- Associated moduli space in the $\mathrm{GL}_2(\mathbf{Q}_p)$ case, and the relation to results of Berger–Breuil.
- Helm–Moss theorem in the $\ell \neq p$ case.
- What we understand in the $\mathrm{GL}_2(\mathbf{Q}_p)$ case — localization of smooth $\mathrm{GL}_2(\mathbf{Q}_p)$-representations, and $D^\natural \boxtimes \mathbf{P}^1$ over the stack.

2. Background reading

The course will be based on [EG19b] and [EG19a], and the introductions to these two papers are a good place to start. The introduction to [CEGS19] may also be helpful.

For background on stacks and representability, we found Artin's original papers on algebraic spaces [Art69b, Art69a, Art70] to be very helpful. The (counter)examples that Artin discusses in [Art69b, §5] illustrate the role of the various hypotheses appearing in Artin's representability theorem. Some of these, along with some additional examples, are also discussed in [EG19b, §4].

We found Colmez's notes `https://webusers.imj-prg.fr/~pierre.colmez/tsinghua.pdf` very helpful when we were trying to understand the Herr complex.

The construction of the stacks was motivated by the Breuil–Mézard conjecture, the weight part of Serre's conjecture, and the relationship between these and the Taylor–Wiles patching method. There will be three afternoon talks about these, but it might still be worth looking at the introduction to [GHS18] for an explanation of our perspective which is hopefully not too obscured by the notation and constructions that tend to proliferate in discussions of patching.

References

[Art69a] M. Artin, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art69b] ______, *The implicit function theorem in algebraic geometry*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 13–34.

[Art70] ______, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. (2) **91** (1970), 88–135.

[CEGS19] Ana Caraiani, Matthew Emerton, Toby Gee, and David Savitt, *Moduli stacks of two-dimensional Galois representations*, arXiv e-prints (2019), arXiv:1908.07019.

[Dri06] Vladimir Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry: an introduction*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 263–304.

[EG19a] Matthew Emerton and Toby Gee, *Moduli stacks of étale* (φ, Γ)*-modules and the existence of crystalline lifts*, arXiv e-prints (2019), arXiv:1908.07185.

[EG19b] Matthew Emerton and Toby Gee, *"Scheme-theoretic images" of certain morphisms of stacks*, Algebraic Geometry (to appear) (2019).

[Eme] Matthew Emerton, *Formal algebraic stacks*, In preparation.

[GHS18] Toby Gee, Florian Herzig, and David Savitt, *General Serre weight conjectures*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 12, 2859–2949.

[PR09] G. Pappas and M. Rapoport, Φ*-modules and coefficient spaces*, Mosc. Math. J. **9** (2009), no. 3, 625–663, back matter.

Email address: `emerton@math.uchicago.edu`

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637

Email address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London, London SW7 2AZ, UK