

BONN LECTURES

MATTHEW EMERTON

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1. LECTURE 1 - INTRODUCTION

These talks are about moduli stacks of (φ, Γ) -modules. In fact, we're really interested in the moduli of (local) (p -adic) Galois representations, so we'll first give some background on this problem, and the complications that arise, and describe how we get to our construction.

Let K be a non-archimedean local field: we care about finite extensions K/\mathbf{Q}_p and we study continuous maps

$$G_K := \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}_d(\overline{\mathbf{F}}_p).$$

Note these factor through $\mathrm{GL}_d(\mathbf{F}_q)$ for some q . Similarly we could look at continuous maps

$$G_K \xrightarrow{\rho} \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$$

and these will factor through $\mathrm{GL}_d(E)$ for some finite E/\mathbf{Q}_p , and using compactness, they actually factor through $\mathrm{GL}_d(\mathcal{O}_E)$.

Example 1.0.1. The first example of an “algebraic family” we might attempt is a family of unramified characters

$$\mathrm{ur}_x : G_K \rightarrow G_K/I_K \cong \mathrm{Frob}^{\widehat{\mathbf{Z}}} \rightarrow \overline{\mathbf{F}}_p^\times$$

which sends Frobenius to $x \in \overline{\mathbf{F}}_p^\times$ (note this is well-defined because any element of $\overline{\mathbf{F}}_p^\times$ lives in \mathbf{F}_q^\times for some q , so we define the unramified character by $\widehat{\mathbf{Z}} \rightarrow \mathbf{Z}/(q-1)\mathbf{Z} \xrightarrow{\sim} \mathbf{F}_q^\times$) But this doesn't quite come

¹Notes taken by Ashwin Iyengar, and have not been looked at or edited by the speaker.

together as an algebraic family: if we wanted to make this algebraic, we would let $\mathbf{F}_p[x, x^{-1}]^\times$ be our ring of coefficients and take specializations to $\overline{\mathbf{F}_p}^\times$. In fact we can define a homomorphism $\text{Frob}^{\mathbf{Z}} \rightarrow \mathbf{F}_p[x, x^{-1}]^\times$ taking $\text{Frob} \mapsto x$, but this doesn't extend continuously to $\text{Frob}^{\widehat{\mathbf{Z}}}$. So this gives the first basic obstruction to constructing the right moduli space.

To get around the problem in the example, you could consider the Weil group, which is a “decompletion” of $\text{Frob}^{\widehat{\mathbf{Z}}}$, but this only works well when $p \neq \ell$, so when $\ell = p$ one has to use (φ, Γ) -modules.

1.1. Weil-Deligne Case. Note the local Galois group G_K has a tame quotient and pro- p wild inertia subgroup:

$$P_K \hookrightarrow G_K \twoheadrightarrow G_K^{\text{tame}} = (\mathbf{Z} \rtimes \mathbf{Z}[1/q])^\wedge$$

where 1 on the left acts by multiplication by q on the right, and we take profinite completion. We can then define the “Weil-Deligne group” WD_K as usual, which fits into the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_K & \longrightarrow & \text{WD}_K & \longrightarrow & \mathbf{Z} \rtimes \mathbf{Z}[1/q] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_K & \longrightarrow & G_K & \longrightarrow & (\mathbf{Z} \rtimes \mathbf{Z}[1/q])^\wedge \longrightarrow 0 \\ & & & & & & \\ & & & & P_K \hookrightarrow W_K \twoheadrightarrow \mathbf{Z} \rtimes \mathbf{Z}[1/q] & & \end{array}$$

Then for any open and finite index subgroup $Q \leq P_K$, WD_K/Q is a finitely presented discrete group, and

$$\text{WD}_K = \varprojlim_Q \text{WD}_K/Q$$

So we have essentially “decompleted and discretized” the tame part, but we are remembering the topology on the wild part of the Galois group. Furthermore, $G_K/Q = (\text{WD}_K/Q)^\wedge$, and since representations of WD_K/Q with values in finite rings do extend over the profinite completion, this seems like a good first approach to defining a moduli space.

Definition 1.1.1. Let $V_Q \rightarrow \text{Spec } \mathbf{Z}$ denote the scheme parametrizing representations $\rho : \text{WD}_K/Q \rightarrow \text{GL}_d$.

Note WD_K/Q is a finitely presented group, so it's easy to find a finite presentation for V_Q as an affine scheme. If $Q' \subseteq Q$, then there is a natural closed immersion (this is easy to check using the moduli description) $V_Q \hookrightarrow V_{Q'}$, so we can study the Ind-scheme

$$V := \varinjlim_Q V_Q$$

By construction, any continuous $\bar{\rho} : G_K \rightarrow \text{GL}_d(\mathbf{F})$ (for \mathbf{F} any finite field) factors through a finite quotient of G_K , and thus a finite quotient of WD_K , and thus factors through one of the WD_K/Q , so gives an \mathbf{F} -valued point of V . Conversely a representation $\text{WD}_K/Q \rightarrow \text{GL}_d(\mathbf{F})$ extends continuously to G_K/Q and thus to G_K . Therefore

$$V(\mathbf{F}) = \{\text{continuous } \bar{\rho} : G_K \rightarrow \text{GL}_d(\mathbf{F})\}.$$

If $R_{\bar{\rho}}^\square$ denotes the universal lifting ring of $\bar{\rho}$ one can check that you get a map

$$\text{Spf } R_{\bar{\rho}}^\square \rightarrow V.$$

In fact, this is versal to V at $\bar{\rho}$. So in some sense, the formal completion of V at each \mathbf{F} -valued point gives you the thickenings of that point. The only problem is that these V_Q are finitely presented affine schemes, but V is a bit more infinitary, so we need to study what sort of geometry we can get when we take the colimit defining V .

So what does V look like? It could involve taking a countable disjoint union of varieties: this is still geometric, and it's locally of finite presentation with countably many components (recall each of the V_Q are

finitely presented). Or it could involve take a variety, then intersecting it with another one, and another one, ad infinitum. This isn't of finite presentation, but it is still a scheme, and it's still quite geometric.

But it could fail to be a scheme! For instance, take $\mathbf{A}_{\mathbf{F}_p}^1$ and glue a copy of $\mathbf{A}_{\mathbf{F}_p}^1$ to each closed point. (WHY ???)

So if V is of one of the first two types, and is still a scheme, then we'd be in good shape because we can do geometry, but otherwise we haven't done much to simplify our original problem. In fact, it really depends on the characteristic of our coefficient field: in characteristic p we will see that things behave badly, while in characteristic 0, we will see that we get a well-behaved formal scheme.

1.2. Analysis of V when $d = 1$ and $K = \mathbf{Q}_p$. For simplicity, assume $p > 2$. In this case, we use local class field theory to replace $G_{\mathbf{Q}_p}$ with

$$G_{\mathbf{Q}_p}^{\text{ab}} = \widehat{\mathbf{Q}_p^\times} = \mathbf{Z}_p^\times \times p^{\mathbf{Z}}$$

Taking the Weil-Deligne subgroup gives

$$\text{WD}_{\mathbf{Q}_p}^{\text{ab}} = \mathbf{Q}_p^\times = \mathbf{Z}_p^\times \times p^{\mathbf{Z}}$$

In this case, the pro- p wild ramification part is a copy of \mathbf{Z}_p , embedded in $\text{WD}_{\mathbf{Q}_p}^{\text{ab}}$ via $1 + p\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p^\times$. So we choose our Q so that

$$\text{WD}_{\mathbf{Q}_p}^{\text{ab}}/Q_n = (\mathbf{Z}/p^{n+1})^\times \times p^{\mathbf{Z}} \cong C_{p^n} \times C_{p-1} \times p^{\mathbf{Z}}$$

Thus

$$V_{Q_n} = \text{Spec}(\mathbf{Z}[x]/(x^{p^n} - 1) \otimes \mathbf{Z}[x]/(x^{p-1} - 1) \otimes \mathbf{Z}[x, x^{-1}]) = \text{Spec} \mathbf{Z}[x]/(x^{p^n} - 1) \times \text{Spec} \mathbf{Z}[x]/(x^{p-1} - 1) \times \mathbf{G}_{m, \mathbf{Z}}$$

- (1) To see what happens away from p , we can invert p on this scheme and see what we get:

$$V_{\mathbf{Z}[1/p]} = \varinjlim_n V_{Q_n, \mathbf{Z}[1/p]} = (\varinjlim_n \text{Spec} \mathbf{Z}[1/p][x]/(x^{p^n} - 1)) \times \text{Spec} \mathbf{Z}[1/p][x]/(x^{p-1} - 1) \times \mathbf{G}_{m, \mathbf{Z}[1/p]}$$

If we now base change to $k = \overline{\mathbf{Q}}$ or $k = \overline{\mathbf{F}_\ell}$ for $\ell \neq p$, then $(x^{p^n} - 1)$ is separable, so we are just adding more and more closed points as we go further along the directed system, and so we basically get infinitely many copies of $\text{Spec} \mathbf{Z}[x]/(x^{p-1} - 1) \times \mathbf{G}_m$, indexed by p th roots of unity in k .

- (2) But instead if we base change to \mathbf{Z}_p , then the closed immersions $V_{Q_n, \mathbf{Z}_p} \hookrightarrow V_{Q_{n+1}, \mathbf{Z}_p}$ are actually nilpotent thickenings, and in fact we end up with

$$V_{\mathbf{Z}_p} = \text{Spf} \mathbf{Z}_p[[T]] \times \text{Spec} \mathbf{Z}[x]/(x^{p-1} - 1) \times \mathbf{G}_m$$

So in this case we get $\text{Spf} \mathbf{Z}_p[[T]]$, which is a nice Noetherian formal scheme: be warned that this is specific to dimension 1: already in dimension 2 the situation becomes more complicated, as we illustrate now.

First, here's a formal scheme that isn't Noetherian. Take $\mathbf{A}_{\mathbf{F}_p}^1$ and add an infinitesimal extra direction at each closed point on the line (one can do this finitely many times and then take a filtered colimit over the finite steps). Then this is the Spf of some topological ring, but this ring will no longer be Noetherian. Essentially we let $\overline{\mathbf{F}_p} = \{a_0, a_1, \dots\}$ and then the ring should be $\varprojlim_j A_j$ where

$$A_j = \overline{\mathbf{F}_p}[x] \times_{\overline{\mathbf{F}_p}} \overline{\mathbf{F}_p}[[x - a_0]] \times_{\overline{\mathbf{F}_p}} \cdots \times_{\overline{\mathbf{F}_p}} \overline{\mathbf{F}_p}[[x - a_j]].$$

Here's something even worse. Take a horizontal line and add infinitely many vertical lines, and then thicken each of the vertical lines (in \mathbf{A}^2) in the horizontal direction. Then this is not even a formal scheme, and it's actually hard to tell this apart from \mathbf{A}^2 (remember we're working over $\overline{\mathbf{F}_p}$).

We could even take the above construction (call it V_\star) and then delete the point at the origin. Keep this in your head for a moment...

1.3. Analysis of V in general. For general d and general K , $V_Q/\mathbf{Z}[1/p]$ is reduced, Cohen-Macaulay, local complete intersection, and flat/ $\mathbf{Z}[1/p]$ of relative dimension d^2 (David Helm) and

$$V_Q/\mathbf{Z}[1/p] \hookrightarrow V_{Q'}/\mathbf{Z}[1/p]$$

is just adding connected components. Note in particular that the local deformation rings of mod ℓ residual representations are just computed using this variety by looking at the complete local rings at the stalks of the corresponding points.

But now let's switch back to the characteristic p setting. Let $K = \mathbf{Q}_p$ and $d = 2$ and $p > 2$ and $1 \leq i \leq p-2$. Let

$$V^{FL,[0,1],\det=\omega^i}/\mathbf{F}_p$$

denote the locus inside V of representations which are Fontaine-Lafaille with Hodge-Tate weights $\{0,1\}$ and with determinant ω^i . Then $V^{FL,[0,1],\det=\omega^i} \cong V_\star$. Note the vertical line whose intersection point with the horizontal line has been deleted corresponds to a family of unramified twists of $\text{Ind}_{\mathbf{Q}_2^p}^{\mathbf{Q}_2^p} \omega_2^i$, and this is disconnected from the rest of the lines... so “the limit of reducible things is reducible”, which is in fact not the behavior that this stack should exhibit.

Note V_\star is very nasty and disconnected, but looks a lot like $\mathbf{A}^2 \setminus \{O\}$, so can we study something else and get \mathbf{A}^2 minus $\{O\}$? Well, you can! If you study families of Fontaine-Lafaille *modules* (the “linear algebra perspective” in p -adic Hodge theory) instead of *representations* then this is exactly what you get.

Over \mathbf{F}_p or \mathbf{Z}_p , V is nasty if $d > 1$, but it sits inside a moduli space of (φ, Γ) -modules which is a lot nicer. Thus we are motivated to define moduli stacks of (φ, Γ) -modules and do a thorough study of their geometry.

2. STACKS, ETC.

There's a tension between rings and topological spaces in algebraic geometry, going back to Diophantus and Descartes. Grothendieck's formulation encompasses both, by giving you a topological space, but also giving you a sheaf of rings. You then solve equations by studying morphisms of schemes.

To study algebraic spaces and stacks, it is helpful to take the functorial point of view, which encompasses the theory of schemes, but gives you a framework to extend it. This perspective starts with the following observation.

Lemma 2.0.1 (Yoneda). *A scheme X can be thought of as a (pre)sheaf $\text{Sch} \xrightarrow{X} \text{Set}$ by taking $Y \mapsto \text{Hom}(Y, X)$. In fact it's a Zariski/étale/fppf sheaf.*

But we can consider more general sheaves. For instance consider a system $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ with closed immersions as transition maps, and define a sheaf

$$X := \varinjlim_i X_i$$

This is an Ind-scheme. Often, this will not actually be a scheme: e.g. embed a point in a line in a plane, in 3-space, etc. This gives you some “infinite dimensional affine space”, which is not a scheme: note the identity map from this space to itself doesn't factor through one of the finite steps, but a map from a quasi-compact scheme does. In our theory, finiteness/quasi-compactness assumptions are used for this purpose.

2.1. Algebraic Spaces. Another example of a more general sheaf is an algebraic space. If X is an fppf sheaf, and U is a scheme and $U \rightarrow X$ is a morphism, then we might say it is “representable by schemes”. This means that in the diagram

$$\begin{array}{ccc} T \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

if T is a scheme then $T \times_X U$ is a scheme. It usually suffices to check this for a certain subclass of T . You could also require that the morphism $U \rightarrow X$ is surjective or étale, which are both properties that one can define via base change to schemes.

Definition 2.1.1. An *algebraic space* is a map $U \rightarrow X$ from a scheme to an fppf sheaf which is representable by schemes, surjective, and étale.

There is an obvious notion of an Ind-algebraic space.

2.2. Formal algebraic spaces. One wants to be able to talk about formal algebraic spaces, and formal algebraic stacks.

Suppose A is a complete topological ring with a countable basis of open neighborhoods of 0 consisting of ideals. Assume that every $A/I_{n+1} \rightarrow A/I_n$ is a nilpotent thickening. Then we define

$$\mathrm{Spf} A = \varinjlim \mathrm{Spec} A/I_n$$

For instance if $A = \mathbf{Z}_p$ and $I_n = p^n$, then we get $\mathrm{Spf} \mathbf{Z}_p$. This is a proper subsheaf of $\mathrm{Spec} \mathbf{Z}_p$. Interestingly, properties like flatness, reducedness, Cohen-Macaulay need the ring A , and can't be seen on the quotients A/I_n . For instance, \mathbf{Z}_p is reduced, while each $\mathbf{Z}/p^n\mathbf{Z}$ for $n > 1$ is not.

Definition 2.2.1. A *formal algebraic space* is a map $\bigsqcup_i U_i \rightarrow X$ where each U_i is an affine formal scheme and X is an fppf sheaf, and such that the map is representable by algebraic spaces, étale and surjective.

Then if $X = \varinjlim X_i$, where X_i are algebraic spaces and the maps are nilpotent thickenings, one can show that X is a formal algebraic space. Vice versa, you can write a formal algebraic space using an Ind-construction.

2.3. Stacks. Stacks are “sheaves” of groupoids. Some (small) groupoids are contractible, and are equivalent to their underlying sets, but some are not: consider the category with one object x and two morphisms $\{1_x, \sigma\}$ where $\sigma^2 = 1_x$: this is not contractible. The moral is that in sets, equality is a *property*, but in higher category theory and the theory of stacks, equality is exhibited by an isomorphism which is some *extra data*.

Really we want to say that a stack is a “2-sheaf”, which means a 2-functor from the category of schemes to the 2-category of groupoids, satisfying a 2-categorical analogue of the sheaf condition. This is technically possible to do, but is a bit complicated once you start trying to work out all the coherence conditions you need, and usually involves making some kind of choices of pullbacks.

On the other hand, one way to avoid this is to use the formalism of *categories fibered in groupoids*, which the stacks project does, and which we will do. For example, a morphism of stacks is a fully faithful embedding of categories fibered in groupoids. An annoying issue is that “isomorphism of stacks” means an equivalence of categories fibered in groupoids.

But there are some interesting things you get from this “extra data”: $\Delta : X \rightarrow X \times X$ is a monomorphism if and only if X is equivalent to something that lands in sets: in other words, it's a usual sheaf of sets. The moral is that the diagonal really tells you something about isomorphisms between objects in the groupoids. For instance, take the pullback along an affine scheme and you should get the isomorphism sheaf between the two objects defined by the morphism from the affine scheme.

2.4. Algebraic Stacks. Let $U \rightarrow X$ be a morphism from a scheme to a stack. It is an **algebraic stack** if it's representable by algebraic spaces, surjective, and smooth. Note that asking for étale is strictly stronger than asking for smooth.

Definition 2.4.1. A map $\bigsqcup_i U_i \rightarrow X$ is a *formal algebraic stack* if it's representable by algebraic spaces, surjective, and smooth. Here U_i are required to be affine formal schemes, and X is an fppf stack.

If X is qcqs, then $X = \varinjlim X_i$, where X_i are algebraic stacks and the transition maps are thickenings. The converse is true if \varinjlim is countably indexed and the transition maps are thickenings.

If $f : X \rightarrow Y$ is a morphism of stacks that is representable by algebraic stacks, you can ascribe geometric properties to f . For instance, if $\Delta : Y \rightarrow Y \times Y$ is representable by algebraic spaces. then any $X \rightarrow Y$ with algebraic stack as the domain is representable by algebraic stacks.

Now imagine X is an algebraic stack, locally of finite type over an excellent locally Noetherian scheme S , and write a map $X \rightarrow F$, where F is limit preserving, and such that Δ_F is representable by algebraic spaces. Since we're representable by algebraic spaces, we can say that $X \rightarrow F$ is proper. In practice, X be will a stack of Breuil-Kisin modules of bounded height, and F will be a stack of étale φ -modules.

In general you wouldn't expect contracting a proper equivalence relation to give you something algebraic. However, we have the following theorem:

Theorem 2.4.2 (Scheme Theoretic Images). *Assume further that F admits effective versal rings at all finite type points, i.e. if we pull back $\mathrm{Spf} R \rightarrow F$ to $X_R \rightarrow \mathrm{Spf} R$ and we take the scheme theoretic image $\mathrm{Spf} S \rightarrow \mathrm{Spf} R$, that this factors through $\mathrm{Spec} S$. Then there exists an algebraic closed substack $Z \hookrightarrow F$ such that $X \rightarrow F$ factors through $X \rightarrow Z$, which is proper and scheme theoretically dominant.*

We will use this theorem repeatedly in our context to construct the Ind-algebraic stacks that we want.

3. HERR COMPLEX

3.1. Herr Complex. Let M be a projective étale (φ, Γ) -module over $\mathbf{A}_{K,A}$. A can be any \mathbf{Z}_p -algebra, but to make nontrivial statements we will often think about the case where A is of finite type over \mathbf{Z}/p^a for some $a \geq 1$. Then we define the *Herr complex*

$$\mathcal{C}^\bullet(M) = [M \xrightarrow{(\varphi-1, \gamma-1)} M \oplus M \xrightarrow{(\gamma-1) \oplus (1-\varphi)} M]$$

concentrated in degrees 0, 1, 2. But note that this is a complex of A -modules, rather than $\mathbf{A}_{K,A}$ -modules (they are individually $\mathbf{A}_{K,A}$ -modules, but the maps are only $\mathbf{A}_{K,A}$ -semilinear).

Theorem 3.1.1. *Suppose A is a Noetherian \mathbf{Z}/p^a -algebra with A/p countable. The Herr complex is a perfect complex of A -modules. If B is a finite type A -algebra, then there is a quasi-isomorphism $\mathcal{C}^\bullet(M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,B}) \simeq B \otimes_A^L \mathcal{C}^\bullet(M)$.*

Proof. Because A is Noetherian, $\mathbf{A}_{K,A}$ is flat over A . Recall M is projective, so M is also flat, so the Herr complex is a complex of flat A -modules (not of finite type over A though, only over $\mathbf{A}_{K,A}$). To check that it is perfect, it suffices to show that the cohomology A -modules are of finite type over A .

If A is finite over \mathbf{Z}/p^a , Herr showed that $\mathcal{C}^\bullet(M)$ computes $H^\bullet(G_K, \rho)$ where ρ is the associated G_K -representation corresponding to M . So the statement we want to prove is sort of like a coefficients version of finiteness of Galois cohomology. But we don't actually have such an equivalence for general coefficient rings. But we have similar behavior:

Lemma 3.1.2. *$H^\bullet(\mathcal{C}^\bullet(M_1^\vee \otimes M_2)) = \mathrm{Ext}_{(\varphi, \Gamma)/\mathbf{A}_{K,A}}^i(M_1, M_2)$ at least for $i = 0, 1$ (and maybe 2). Also $H^2(\mathcal{C}^\bullet(\mathrm{ad} M))$ contains the obstruction classes to infinitesimal deformations of M .*

We have a map $\psi : \mathbf{A}_{K,A} \rightarrow \mathbf{A}_{K,A}$ and $\psi : M \rightarrow M$, which is defined so that $\psi(\varphi(a)m) = a\psi(m)$ and $\psi(a\varphi(m)) = \psi(a)m$, so we roughly think of $\psi = 1/p \times \mathrm{tr} \varphi$.

Then $1 - \gamma : M^{\psi=0} \rightarrow M^{\psi=0}$ is an isomorphism. Explanation just for $K = \mathbf{Q}_p$ using $\tilde{\Gamma}$ and $M = \mathbf{A}'_{\mathbf{Q}_p}$: then $\mathbf{Z}_p[[T]] = \mathbf{Z}_p[[\mathbf{Z}_p]]$, which has a natural action of \mathbf{Z}_p^\times . Therefore $\mathbf{Z}_p[[\mathbf{Z}_p]]^{\psi=0} = \mathbf{Z}_p[[\mathbf{Z}_p^\times]] = \mathbf{Z}_p[[\Delta]] \otimes \mathbf{Z}_p[[x]]$, where $x = \gamma - 1$.

Now

$$\begin{array}{ccccccc}
 \mathcal{C}^\bullet(M) : & 0 & \longrightarrow & M \xrightarrow{(\varphi^{-1}, \gamma^{-1})} M \oplus M \xrightarrow{\gamma^{-1} \oplus 1 - \varphi} M & \longrightarrow & 0 \\
 & & & \parallel & \downarrow -\psi \otimes \text{id} & \downarrow -\psi \\
 \mathcal{C}_\psi^\bullet(M) : & 0 & \longrightarrow & M \xrightarrow{(\psi^{-1}, \gamma^{-1})} M \oplus M \xrightarrow{\gamma^{-1} \oplus 1 - \psi} M & \longrightarrow & 0
 \end{array}$$

These are quasi-isomorphic. Note A is countable and $\mathbf{A}_{K,A}$ has a countable basis, so we can use the theory of Polish groups. The top is discrete, the bottom is finitely generated, so we compare the two perspectives on cohomology. \square

3.2. Families of G_K -reps into $\text{GL}_d(\overline{\mathbf{F}}_p)$. Start with $K = \mathbf{Q}_p$ and $d = 1$. Then by local class field theory the $\overline{\mathbf{F}}_p^\times$ -valued characters of G_K are all of the form $\omega^i \text{ur}_\alpha$, for $i = 0, \dots, p-2$ and $\alpha \in \overline{\mathbf{F}}_p^\times$. So if I take the trivial (φ, Γ) -module over $\mathbf{A}'_{\mathbf{Q}_p, \mathbf{F}_p[\alpha, \alpha^{-1}]}$ and twist φ by α and $\tilde{\Gamma}$ by ω^i , we get a \mathbf{G}_m worth of representations, but also a \mathbf{G}_m worth of scalar automorphisms. In summary we obtain a map

$$\bigsqcup_{i=0}^{p-2} [\mathbf{G}_m / \mathbf{G}_m] \rightarrow (\mathcal{X}_1)_{\text{red}}$$

which is actually an isomorphism.

Now let $d = 2$. If you're irreducible, then you are of the form (for some i as above)

$$\text{ur}_\alpha \text{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p} \omega_2^i$$

This gives us a point $[\mathbf{G}_m / \mathbf{G}_m] \hookrightarrow (\mathcal{X}_2)_{\text{red}}$ (again the α gives a \mathbf{G}_m and then we quotient by scalars). But we know from our dimension computation that $\dim(\mathcal{X}_2)_{\text{red}} = 1$ and that it's equidimensional. So these don't exhaust the whole space: we have reducible characters to factor in. So we can look at things in $\text{Ext}^1(\text{ur}_\alpha \omega^i, \text{ur}_\beta \omega^j)$. But there's a unique nontrivial extension, so the intuition is that as α, β vary, the extension class varies with them, and then you mod out by scalars to get something one dimensional.

In fact, we want to do this same technique in more generality. The point is to start with a representation and then iteratively build its space of extensions by some irreducible representation. More generally, assume that we're given a family $\bar{\rho}_T \rightarrow T$ of rank d étale (φ, Γ) -modules, where T is a reduced finite type variety over $\overline{\mathbf{F}}_p$. Then take $\bar{\sigma}$ an irreducible Galois representation over T , and suppose $\text{Ext}_{G_K}^2(\bar{\sigma}, \bar{\rho}_t)$ are of constant dimension when we vary over t .

Note our base change property for $\mathcal{C}^\bullet(M)$ involves a derived tensor product, so there's some spectral sequence relating $\mathcal{C}^\bullet(M)$ and $\mathcal{C}^\bullet(M \otimes_A B)$, but since H^2 is the highest degree, it satisfies naive base change.

Therefore, $H^2(\mathcal{C}^\bullet(\bar{\sigma}^\vee \otimes \bar{\rho}_T))$ is actually a vector bundle over T (because the Ext^2 have constant degree), which implies that $\tau_{\leq 1} \mathcal{C}^\bullet(\bar{\sigma}^\vee \otimes \bar{\rho}_T)$ is still perfect, and can thus be modeled as $[C^0 \rightarrow Z^1]$ concentrated in degrees 0, 1, which are locally free over T . Note $Z^1 \rightarrow H^1(\mathcal{C}^\bullet(\bar{\sigma}^\vee \otimes \bar{\rho}_T)) = \text{Ext}^1(\bar{\sigma}, \bar{\rho}_T)$. So let V be the vector bundle over T corresponding to Z_1 . Let $\bar{\rho}_V$ be the pullback of $\bar{\rho}_T$ to V . By analogy with the 2-dimensional case, we want to study a "universal extension" \mathcal{E}_V of $\bar{\rho}_V$ by $\bar{\sigma}$, which should fit in the diagram

$$0 \rightarrow \bar{\rho}_V \rightarrow \mathcal{E}_V \rightarrow \bar{\sigma} \rightarrow 0$$

To construct this thing, we write

$$\text{Ext}^1(\bar{\sigma}, \bar{\rho}_T \otimes (Z_1)^\vee) = \text{Ext}^1(\bar{\sigma}, \bar{\rho}_T) \otimes (Z_1)^\vee$$

which is a quotient of

$$Z^1 \otimes (Z^1)^\vee$$

which has a "trace element", which maps to the universal extension. So

$$0 \rightarrow \bar{\rho}_T \otimes (Z^1)^\vee \rightarrow \text{ext} \rightarrow \bar{\sigma} \rightarrow 0$$

giving

$$0 \rightarrow \bar{\rho}_T \otimes \mathcal{O}_V \rightarrow \mathcal{E}_V \rightarrow \bar{\sigma} \rightarrow 0$$

The point is that this will let us describe the underlying reduced stack in general, which we'll return to in the next lecture.

4. EXTENSIONS AND CRYSTALLINE LIFTS

So intuitively, a point of V is a point of T and a vector over it; the point should give us a mod p Galois representation, and the vector specifies an extension class. So \mathcal{E}_V reflects this intuition, and glues all of these things together.

So again return to the $K = \mathbf{Q}_p$ and $d = 2$ case. Take $T = \mathbf{G}_m = \text{Spec } \mathbf{F}_p[\alpha, \alpha^{-1}]$ and $\bar{\rho}_T = \text{ur}_\alpha \omega^i$. For the moment let $\bar{\sigma} = 1$ (this will suffice to describe all the geometric phenomena, then we can twist by ω to get the remaining cases). Then

$$\text{Ext}^2(1, \text{ur}_\alpha \omega^i) = H^2(G_{\mathbf{Q}_p}, \text{ur}_\alpha \omega^i)$$

which is 0 unless $i = 1$ and $\alpha = 1$, in which case it's 1-dimensional.

- When $i = 2, \dots, p-2$,

$$\text{Ext}^2(1, \text{ur}_\alpha \omega^i) = 0,$$

and

$$\text{Ext}^0(1, \text{ur}_\alpha \omega^i) = (\text{ur}_\alpha \omega^i)^{G_{\mathbf{Q}_p}} = 0$$

so actually (by local Tate duality) $\text{Ext}^1(1, \text{ur}_\alpha \omega^i)$ has constant rank 1, so $\text{Ext}^1(1, \bar{\rho}_T)$ is a line bundle. Its total space parametrizes extensions

$$\begin{pmatrix} 1 & * \\ 0 & \text{ur}_\alpha \omega^i \end{pmatrix}$$

and these have only scalar automorphisms. Therefore, we get a family in $(\mathcal{X}_2)_{\text{red}}$ that looks like

$$[\mathbf{A}^1 \times \mathbf{G}_m / \mathbf{G}_m]$$

which is visibly 1-dimensional.

- When $i = 0$, local Tate duality gives us that

$$\text{Ext}^2(1, \text{ur}_\alpha) = H^2(G_{\mathbf{Q}_p}, \text{ur}_\alpha) = H^0(G_{\mathbf{Q}_p}, \text{ur}_\alpha \otimes \epsilon^{-1}) = 0$$

for all $\alpha \in \overline{\mathbf{F}_p}^\times$, but now we run into the issue where Ext^0 is not a vector bundle, because it has rank 0 everywhere except for when $\alpha = 1$, and at this point the rank jumps from 0 to 1. What this means is that we get a family in $(\mathcal{X}_2)_{\text{red}}$ which looks like a sort of weird cusp-y thing (???)

- When $i = 1$, local Tate duality gives

$$\dim \text{Ext}^2(1, \text{ur}_\alpha \omega) = \dim H^2(G_{\mathbf{Q}_p}, \text{ur}_\alpha \omega) = \dim H^0(G_{\mathbf{Q}_p}, \text{ur}_\alpha) \cong \begin{cases} 1 & \alpha = 1 \\ 0 & \alpha \neq 1 \end{cases}$$

so Ext^2 is no longer a vector bundle. Note

$$H^0(G_{\mathbf{Q}_p}, \text{ur}_\alpha \omega) = 0.$$

Geometrically, this means that on $\mathbf{G}_m \setminus \{1\}$, we get $[\mathbf{A}^1 \times (\mathbf{G}_m \setminus \{1\}) / \mathbf{G}_m]$ and on $\{1\}$ we get $[\mathbf{A}^2 / \mathbf{G}_m]$. So in the end, the pictures fit together at $\alpha = 1$ and we get two planes intersecting each other transversally in a line.

Remark 4.0.1. One has to be careful: in the above analysis remember that the stack \mathcal{X}_2 is really a stack of (φ, Γ) -modules, so to literally construct the geometric objects above, one has to construct a (φ, Γ) -module in some ring of coefficients, and then view it as a point of the underlying reduced stack $(\mathcal{X}_2)_{\text{red}}$. For the purposes of gaining intuition, the above is a good sketch of what's going on.

We can label these components by Serre weights (the duals are an artifact of a choice of ordering earlier):

- When $i = 2, \dots, p - 2$, we assign the label $(\mathrm{Sym}^{i-1} \mathbf{F}_p^2)^\vee$.
- When $i = 0$, we assign the label $(\mathrm{Sym}^{p-2} \mathbf{F}_p^2)^\vee$
- When $i = 1$ we assign the label $(\mathrm{Sym}^{p-1} \mathbf{F}_p^2)^\vee$.

Actually, using this construction technique inductively, we can prove the following general theorem.

Theorem 4.0.2.

$$(\mathcal{X}_d)_{\mathrm{red}} = \left(\bigcup_{\text{Serre weights}} \text{closure of an irred. niveau 1 family of dim } [K : \mathbf{Q}_p]d(d-1)/2 \right) \cup \text{lower dim things}$$

In fact, there are no lower dimensional things, so really

$$(\mathcal{X}_d)_{\mathrm{red}} = \bigcup_{\text{Serre weights}} \text{closure of an irred. niveau 1 family of dim } [K : \mathbf{Q}_p]d(d-1)/2$$

The dimension argument (i.e. showing that there are no smaller dimensional components in the reduced substack) relies on the existence of crystalline lifts. We use the fact that the crystalline stack is p -adic to deduce that if X is a component of a crystalline deformation ring, then the codimension of

$$\{x \in X \mid H^2(\rho_x) \geq r\}$$

(which is Zariski closed) is at least $r + 1$.

Example 4.0.3. Take $\bar{\rho} = \omega$ and $X = \mathrm{Spec} R_{\bar{\rho}}^{\square, \mathrm{cris}, HTwt=-p} = \mathrm{Spec} \mathbf{Z}_p[[\alpha - 1]]$, over which lives the family $\mathrm{ur}_\alpha \epsilon^p$. Then H^2 is supported on the locus $(\alpha - 1, p)$, where it has dimension 1.

Theorem 4.0.4. If $\bar{\rho} : G_K \rightarrow \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ is continuous, then there exists $\rho^\circ : G_K \rightarrow \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ lifting $\bar{\rho}$ such that the corresponding p -adic Galois representation $\rho : G_K \rightarrow \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ is crystalline with regular Hodge-Tate weights, and potentially diagonalizable (which implies, if $p \nmid 2d$, then $\bar{\rho}$ can be globalized to come from an automorphic form).

Theorem 4.0.5. Suppose we're given $0 \rightarrow \bar{\rho}_d \rightarrow \bar{\rho}_{d+a} \rightarrow \bar{\sigma} \rightarrow 0$ where $\bar{\sigma}$ is irreducible of dimension a and suppose X is a non-empty component of a crystalline deformation ring for $\bar{\rho}_d$. Let σ be a crystalline lift of $\bar{\sigma}$ chosen so that all extensions of σ by points of X are crystalline: in other words, you need to choose the Hodge-Tate weights of σ to be suitably spaced with respect to the Hodge-Tate weights of $x \in X$.

Then there exists some $\rho_d^0 \in X(\overline{\mathbf{Z}}_p)$ such that there exists an extension

$$0 \rightarrow \rho_d^\circ \rightarrow \rho_{d+a}^\circ \rightarrow \sigma \rightarrow 0$$

lifting the original sequence.

5. MODULI SPACES AND BERNSTEIN CENTERS

“Moduli spaces” refers to the spaces associated to moduli stacks.

An algebraic stack \mathcal{X} has an underlying “Zariski” topological space with points, formed by taking a smooth cover by a scheme, taking its topological space, and then taking a quotient topological space. But you can look for a map $f : \mathcal{X} \rightarrow X$ to X an algebraic space which is the “best possible approximation to the stack” and has no automorphisms. Loosely speaking, one might call X a *moduli space associated to \mathcal{X}* if f is initial in the space of maps to algebraic spaces. Or say f is an *associated moduli space* morphism.

But this doesn't really help you access X or say anything concrete about it. But Jared Alper gives properties that ensure that f is initial for morphisms to locally separated algebraic spaces (for instance, a line with the origin replaced by a \mathbf{P}^1 is not locally separated). The characterization:

- (1) $f^* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$
- (2) f is a universal submersion (“surjective, induces quotient topology”)
- (3) If k is algebraically closed, $\mathcal{X}(k) \rightarrow X(k)$ is given by the equivalence relation generated by $x \sim y$ if $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$.

Think about $\mathcal{X} = [\mathbf{A}^1/\mathbf{G}_m]$. Then there are two k -points: the origin, and a fuzzy one which specializes to the origin, so the associated moduli space is just a point.

Keel-Mori (plus various technical improvements) implies that if \mathcal{X} is Deligne-Mumford (and maybe quasi-DM?) then there exists X which is a “coarse moduli space” (the maps $\mathcal{X} \rightarrow X$ induces a bijection on k -points).

Then Alper has a theory of “good” and “adequate” moduli spaces. One feature of this theory is that if $f : \mathcal{X} \rightarrow X$ is good or adequate, then it is universally closed. Also, all stabilizers at closed k -points are reductive. Also, the relation $x \sim y$ if $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$ is an equivalence relation (under some assumption... good/adequate?).

Example 5.0.1. If G is reductive over k and A is a finite type k -algebra with a G -action, then $[\mathrm{Spec} A/G] \rightarrow \mathrm{Spec} A^G$ is adequate (or good in characteristic 0).

Take \mathbf{P}^1 , which has a \mathbf{G}_m -action, and take $[\mathbf{P}^1/\mathbf{G}_m]$. Similar to the affine case there's a big open point, but two closed points, and the open point specializes to both.

In Galois deformation theory, the passing from Galois representations to pseudorepresentations is an example of this theory we're discussing. One subtlety is that traces don't know about extension classes, so this sort of factors through semisimplification. So the closed points of $(\mathcal{X}_d)_{\mathrm{red}}(\overline{\mathbf{F}}_p)$ correspond to semisimple representations. Specialization corresponds to semisimplification. In other words, $\overline{\{\rho\}}$ has a unique closed point, which is $\overline{\rho}^{ss}$.

In our case, stabilizers of closed points are reductive, and the $x \sim y$ is an equivalence relation, but our map is not universally closed, so we don't have adequate moduli spaces.

For instance look at $[\mathbf{A}^2 \setminus \{0\}/\mathbf{G}_m]$. Then the vertical lines specialize to their intersection with the horizontal axis, away from the origin. The vertical line at the origin is closed. There's an obvious map to \mathbf{A}^1 , which is projection to the horizontal axis, which IS the associated moduli space map: but on the other hand, it's *not* adequate, for instance because it's not closed.

Nevertheless, we expect the following.

Theorem 5.0.2. *Let $K = \mathbf{Q}_p$. Then $(\mathcal{X}_2^{\det=x})_{\mathrm{red}}$ admits a moduli space which is a finite chain of \mathbf{P}^1 s. Its complete local rings are pseudodeformation rings.*

We expect that \mathcal{X}_2 admits an associated formal moduli space which would be a thickening of this.

How does this picture relate to crystalline deformation rings? Given a crystalline representation $V_{a_p, k}$, then $|a_p| \leq 1$. So the rigid generic fiber of the moduli space is just this disk, and for the stack you get a formal model, lands in the \mathbf{P}^1 s? The easiest model is $\widehat{\mathbf{A}}^1_{\mathbf{Z}_p}$, but if you blow up in the special fiber, you get \mathbf{P}^1 s.

5.1. Bernstein Centers. It makes sense to connect the Hecke side to pseudocharacters, since you get numbers from Hecke eigenvalues. So for $\ell \neq p$ we had $\mathcal{V}_Q = [\mathrm{Spec} A_Q / \mathrm{GL}_d]$ and the associated moduli space is just $\mathrm{Spec} A_Q^{\mathrm{GL}_d}$.

Theorem 5.1.1 (Helm-Moss). $\varprojlim_Q A_Q^{\mathrm{GL}_d}$ is the \mathbf{Z}_ℓ -Bernstein center for $\mathrm{GL}_d(K)$.

This is supposed to encompass the step in automorphy lifting theorems where you match characters on the Galois and automorphic sides.

Theorem 5.1.2 (Dotto-Emerton-Gee). *If \mathcal{A} is the abelian category of smooth $\mathrm{GL}_2(\mathbf{Q}_p)$ -representations on \mathbf{Z}_p -modules which are locally p -power torsion, then \mathcal{A} localizes to a stack of categories over the chain of \mathbf{P}^1 s.*

Then our expectation is that \mathcal{O}_{X_2} is the sheaf of Bernstein centers of \mathcal{A} . So what's the identification?

$D \rightarrow \mathcal{X}_2^{\mathrm{det}^x}$ is the universal (φ, Γ) -module. Then $D \boxtimes \mathbf{P}^1 / D^\natural \boxtimes \mathbf{P}^1$ should be a quasi-coherent sheaf of $\mathrm{GL}_2(\mathbf{Q}_p)$ -representations over \mathcal{X}_2 (mod finite dimensional representations).

Then the equality is governed by saying that the two sheaves of rings act in the same way on this family. This is strongly related to, and uses, Paskunas's work.