A FREE BOUNDARY PROBLEM FOR AN ELLIPTIC SYSTEM

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Abstract. We study solutions and the free boundary $\partial\{|\mathbf{u}| > 0\}$ of the sublinear system

$$\Delta\mathbf{u} = \lambda_+(x)|\mathbf{u}^+|^{q-1}\mathbf{u}^+ - \lambda_-(x)|\mathbf{u}^-|^{q-1}\mathbf{u}^-,$$

from a regularity point of view.

For $\lambda_\pm(x) > 0$ and Hölder, and $0 < q < 1$, we apply the epiperimetric inequality approach and show $C^{1,\beta}$-regularity for the free boundary at asymptotically flat points.

Contents

1. Introduction

1.1. **Problem setting.** In this paper we study the elliptic system

$$\Delta\mathbf{u} = \lambda_+(x)|\mathbf{u}^+|^{q-1}\mathbf{u}^+ - \lambda_-(x)|\mathbf{u}^-|^{q-1}\mathbf{u}^-, \tag{1}$$

where $\lambda_\pm > 0$ are Hölder regular, $\mathbf{u} = (u_1, \ldots, u_m)$, with $\mathbf{u} : B_1 \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $n \geq 2$, $m \geq 1$, and $\mathbf{u}^\pm = (u_1^\pm, \cdots, u_m^\pm)$. Here $|\cdot|$ stands for the Euclidian norm, $B_1 = B_1(0)$ is the unit ball, and the equation is in the weak sense. Solutions of (1) are the unique minimizers (up to the prescribed boundary values) of the energy

$$J_0(\mathbf{u}) = \int_{B_1} \left(|\nabla\mathbf{u}|^2 + \frac{2}{1+q}\lambda_+(x)|\mathbf{u}^+|^{1+q} + \frac{2}{1+q}\lambda_-(x)|\mathbf{u}^-|^{1+q} \right) dx. \tag{2}$$

We are interested in the regularity of both minimizers $\mathbf{u}$ of (2) and their free boundaries $\partial\{x : |\mathbf{u}(x)| > 0\}$. Our departing point is a $W^{1,2}$-solution to this equation, regardless of the boundary data. Since for each $i = 1, \cdots, m$, $\Delta u_i \in \mathcal{L}^{2/q}$ we will have $u_i \in W^{2,2/q}$ and a bootstrap argument will show that $u_i \in W^{2,p}$ for all $p < \infty$. The main question is about the higher regularity of the solution along with the regularity of the free boundary $\partial\{|\mathbf{u}| > 0\}$.

For clarity of exposition, and for readers' convenience we shall carry out the analysis for the case $\lambda_+ = \lambda_- \equiv 1$. In Section 5 we shall explain the obvious and necessary changes for the general case in (1). We thus, in what follows, consider the equation

$$\Delta \mathbf{u} = f(\mathbf{u}) := |\mathbf{u}|^{q-1}\mathbf{u}, \qquad \text{in } B_1(0), \qquad \text{where } q \in (0,1), \tag{3}$$

that are minimizers to

$$J(\mathbf{u}) = \int_{B_1} \left(|\nabla \mathbf{u}|^2 + 2F(\mathbf{u})\right) dx, \tag{4}$$

where

$$F(\mathbf{u}) = \frac{1}{1+q}|\mathbf{u}|^{1+q}.$$

The case $q = 0$ was studied in [2] and it has been shown that the set of "regular"[1] free boundary points is locally a $C^{1,\beta}$ surface. It is noteworthy that in the scalar case, when $m = 1$, one recovers also the two phase free boundary problem

$$\Delta u = (u^+)^q - (u^-)^q, \tag{5}$$

which was investigated in [4], from a regularity point of view, with partial results. When solutions of (5) are assumed to be non-negative, the optimal regularity $C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}$ for the solution has been shown, where $\kappa = 2/(1-q)$, as well as the regularity of the free boundary close to almost flat points; see [1, 3, 7, 8].

In this paper we study the behaviour of solutions as well as the free boundary close to asymptotically flat points, and obtain results along the lines of [2].

1.2. **Notations and Definitions.** For clarity of exposition we shall introduce some notation and definitions here that are used frequently in the paper.

Throughout this paper, $\mathbb{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, $B_r(x_0)$ will denote the open n-dimensional ball of center x_0, radius r with the boundary $\partial B_r(x_0)$. In addition, $B_r = B_r(0)$ and $\partial B_r = \partial B_r(0)$. For a set A, $d(x, A)$ stands for the distance between x and A. In the text we use the n-dimensional Hausdorff measure $\mathcal{H}^n$. For a real number s, we denote the greatest integer below s by $\lfloor s \rfloor$, i.e. $s - 1 \le \lfloor s \rfloor < s$.

Also, we will denote the derivative of function f by $f_{\mathbf{u}}$ and the derivative matrix of $\mathbf{u}$ by $\nabla \mathbf{u} = [\partial_i u_j]_{1\le i\le n, 1\le j\le m}$ with the notation

$$|\nabla \mathbf{u}|^2 = \sum_{i=1}^m |\nabla u_i|^2, \qquad \nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^m (\nabla u_i \cdot \nabla v_i),$$
$$\nabla \mathbf{u} \cdot \xi = \xi^t \nabla \mathbf{u} = (\nabla u_1 \cdot \xi, \cdots, \nabla u_m \cdot \xi), \qquad \text{for all } \xi \in \mathbb{R}^n.$$

We denote by $\Gamma(\mathbf{u}) = \partial\{|\mathbf{u}| > 0\} \cap \{|\nabla \mathbf{u}| = 0\}$ the set of free boundary. Moreover, for $q \in (0,1)$ we fix the following constants throughout the paper:

$$\kappa = \frac{2}{1-q}, \qquad \alpha = (\kappa(\kappa-1))^{-\kappa/2}.$$

[1] For the meaning of regular points, see Definition 1.3.

1.3. **Main results.** Let us first assume that a solution $\mathbf{u} = (u_1, \dots, u_m)$ of (3), is such that all components of $\mathbf{u}$ are positive. Then we have the following result.

Proposition 1.1 (Regularity near the one-phase free boundary points)**.** *Let* $\mathbf{u}$ *be a solution of the system* (3)*, and* $u_i \geq 0$ *in* $B_r(x_0)$ *for some* i*. Then there is a constant* $c = c(n, q)$ *such that*

$$u_i(x) \leq c(u_i(x_0) + |x - x_0|^\kappa), \quad \forall x \in B_{r/2}(x_0).$$

This theorem shows that if $\mathbf{u}(x_0) = 0$, then all derivatives of $\mathbf{u}$ of order less than κ at point x_0 vanish. However, we can not expect to obtain $C^{\lfloor\kappa\rfloor, \kappa - \lfloor\kappa\rfloor}$-regularity in the general case, particularly when some components of $\mathbf{u}$ change signs. Indeed, the ODE $y'' = y^q$, with initial condition $y(0) = 0 \neq y'(0)$, has a solution whose third derivative is unbounded, $y''' = qy'y^{q-1}$.

In order to study the optimal decay of solutions near such points, we start with a definition of the subset $\Gamma^s(\mathbf{u})$ of the free boundary $\Gamma(\mathbf{u})$ as follows

$$\Gamma^s(\mathbf{u}) := \Big\{ z \in \Gamma(\mathbf{u}) : \text{there exists some } c > 0 \text{ and a vector function } \mathbf{P}_m \text{ that each component is a polynomial of degree at most } m < s, \text{ such that for all } r > 0 \text{ we have } \sup_{B_r(0)} |\mathbf{u}(x+z) - \mathbf{P}_m(x)| \leq cr^s \Big\}.$$

We will show that $\Gamma^\kappa(\mathbf{u})$ contains only points at that all derivatives of order less than κ are zero.

Theorem 1.2. *Let* $\mathbf{u}$ *be a solution of the system* (3) *with* $\mathbf{u}(z) = 0$*. Consider* $\ell = \lfloor\kappa\rfloor$ *to be the greatest integer below* κ*, i.e.* $\kappa - 1 \leq \ell < \kappa$*, then* $z \in \Gamma^\kappa(\mathbf{u})$ *if and only if*

$$\sup_{B_r(z)} |\mathbf{u}| = o(r^\ell).$$

To investigate the regularity of free boundary, we consider "asymptotically one-phase-points" that is, a subset of $\Gamma(\mathbf{u})$ such that the blow-ups belong to

$$\mathbb{H} := \{x \mapsto \alpha \max(x \cdot \nu, 0)^\kappa \mathbf{e} : \nu \in \mathbb{R}^n \text{ and } \mathbf{e} \in \mathbb{R}^m \text{ are unit vectors}\}.$$

Members of these class are κ-homogeneous global solutions of (3). When the domain is a plane, i.e. $n = 2$, all κ-homogeneous global solutions are classified (see Proposition 6.1 in Appendix). The members of $\mathbb{H}$ are called *half-plane* solutions.

Definition 1.3. *We denote by* $\mathcal{R}_\mathbf{u}$ *the set of all regular free boundary points of* $x_0 \in \Gamma(\mathbf{u})$*, which has at least one blow-up limit of* $\mathbf{u}$ *at* x_0 *that belongs to* $\mathbb{H}$*.*

This definition is well-defined according to the uniqueness of blow-up, as we will show later. (See Remark 4.4). Our main result concerning the regularity of the free boundary is presented in the following theorem.

Theorem 1.4 (Regularity of the free boundary)**.** *The set of regular free boundary points* $\mathcal{R}_\mathbf{u}$ *is locally in* B_1 *a* $C^{1,\beta}$*-manifold.*

We choose the epiperimetric inequality approach to prove this result. Since the first application of this approach in [9], it has been used in various articles (see for example [2] for an application in a system or [10] for a sublinear scalar equation case). This inequality, Theorem 3.1, with a monotonicity formula, Proposition 2.3, provides an estimate for the energy decay. Indeed, one can control the rate of convergence $\|\mathbf{u}(x_0 + \cdot) - \mathbf{h}\|_{\mathcal{L}^1(B_r)}$ in this approach, where $\mathbf{h}$ belongs to $\mathbb{H}$. In

Theorem 4.7, we will show that when $\mathbf{h} \in \mathbb{H}$, the rate of convergence is $r^{n+\kappa+\beta}$ for some $\beta > 0$.

In order to keep the presentation simple, we consider the constant coefficient case (3) and do all calculation first for that. In Section 5, the result is extended to the general form (1).

2. Higher Regularity of Solutions

In this section we will study the regularity of solutions of (3) and prove Proposition 1.1 and Theorem 1.2.

Proof of Proposition 1.1. Let $\varphi(r) := \fint_{\partial B_r(x_0)} u_i$ and observe that $u_i \geq 0$, in the statement of the theorem. Since $\Delta u_i = |\mathbf{u}|^{q-1} u_i \geq 0$, we know that $\varphi(r)$ is increasing and

$$\varphi'(r) = \frac{r}{n}\fint_{B_r(x_0)} |\mathbf{u}|^{q-1} u_i \leq \frac{r}{n}\Big(\fint_{B_r(x_0)} u_i\Big)^q \Big(\fint_{B_r(x_0)} \frac{u_i}{|\mathbf{u}|}\Big)^{1-q} \leq \frac{r}{n}\big(\varphi(r)\big)^q, \tag{6}$$

where in the last inequality we have applied $\fint_{B_r(x_0)} u_i \leq \varphi(r)$. Form (6), we obtain

$$\varphi(r)^{1-q} - \varphi(0)^{1-q} \leq \frac{r^2}{2n},$$

and hence

$$\varphi(r) \leq (u_i(x_0)^{1-q} + \frac{r^2}{2n})^{\kappa/2} \leq C_q(u_i(x_0) + r^\kappa).$$

On the other hand, u_i is a nonnegative subharmonic function and there is a constant C_n such that

$$u_i(x) \leq C_n \fint_{\partial B_\rho(x_0)} u_i, \quad \text{for every } x \in B_{r/2}(x_0) \text{ and } \rho = 2|x - x_0|.$$

Therefore, $u_i(x) \leq c(n,q)(u_i(x_0) + \rho^\kappa) \leq c(u_i(x_0) + |x - x_0|^\kappa)$. □

Corollary 2.1. *Let* $\mathbf{u}$ *be a solution of the system* (3) *and* $x_0 \in \Gamma$. *There exists a constant* $c = c(n,q)$ *such that if* $u_i \geq 0$ *in* $B_r(x_0)$ *for all* i, *then*

$$\sup_{B_r(x_0)} |\mathbf{u}| \leq c r^\kappa.$$

Now we are going to prove Theorem 1.2. The necessity of vanishing derivatives is deduced from the following proposition when $s = \kappa$. The proof follows the same line of reasoning as that of the proof of Proposition 2.1 in [4].

Proposition 2.2. *Let* $\mathbf{u}$ *be a solution of system* (3) *and* $z \in \Gamma^s(\mathbf{u})$ *for* $s \leq \kappa$, *then all the derivatives of order* $m < \frac{s-2}{q}$ *at point* z *are zero.*

This proposition in a special case will imply that if $z \in \Gamma^\kappa(\mathbf{u})$ then all derivatives of $\mathbf{u}$ at the point z up to order $m < \kappa$ exist and are equal to zero, i.e. we must have $\mathbf{P}_m(x) = 0$ in definition of $\Gamma^\kappa(\mathbf{u})$. Thus when we are looking for points with regularity $C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}$ in the free boundary $\Gamma(\mathbf{u})$, we might find them among the points where all the derivatives below κ exist and are zero. Theorem 1.2 shows that this is a sufficient condition for a free boundary point to belong to $\Gamma^\kappa(\mathbf{u})$. We divide the proof in two different cases, depending on whether κ is integer or not. Before that we need to show that the monotonicity formula (which is established by the third author in [9] for the classical obstacle problem), holds in the present setting.

Proposition 2.3. *Let* $\mathbf{u}$ *be a solution of* (3) *in* $B_{r_0}(x_0)$ *and let*

$$W(\mathbf{u},x_0,r)=\frac{1}{r^{n+2\kappa-2}}\int_{B_r(x_0)}\left(|\nabla\mathbf{u}|^2+2F(\mathbf{u})\right)dx-\frac{\kappa}{r^{n+2\kappa-1}}\int_{\partial B_r(x_0)}|\mathbf{u}|^2\,d\mathcal{H}^{n-1}.$$

(i) For $0<r<r_0$*, the energy function* $W(\mathbf{u},x_0,r)$ *is non-decreasing.*
(ii) The function $x\mapsto W(\mathbf{u},x,0+)$ *is upper-semicontinuous.*

Proof. For $\mathbf{u}_r(x):=\mathbf{u}(x_0+rx)/r^\kappa$ we can apply the relations $r\partial_r\mathbf{u}_r=\nabla\mathbf{u}_r\cdot x-\kappa\mathbf{u}_r$ and $W(\mathbf{u}_r,0,s)=W(\mathbf{u},x_0,rs)$ to write for $s>t>0$,

$$W(\mathbf{u},x_0,s)-W(\mathbf{u},x_0,t)=\int_t^s\int_{\partial B_1(0)}\frac{2}{r}|\nabla\mathbf{u}_r\cdot x-\kappa\mathbf{u}_r|^2d\mathcal{H}^{n-1}dr\geq 0.$$

For *(ii)*, if $W(\mathbf{u},x_0,0+)>-\infty$, for an arbitrary $\varepsilon>0$, we may by monotonicity, part (i), choose r such that $W(\mathbf{u},x_0,r)\leq W(\mathbf{u},x_0,0+)+\varepsilon/2$. For this fixed r, there is a δ-neighborhood of x_0 such that $W(\mathbf{u},x,r)\leq W(\mathbf{u},x_0,r)+\varepsilon/2$. Therefore,

$$W(\mathbf{u},x,0+)\leq W(\mathbf{u},x,r)\leq W(\mathbf{u},x_0,r)+\varepsilon/2\leq W(\mathbf{u},x_0,0+)+\varepsilon.$$

The case, $W(\mathbf{u},x_0,0+)=-\infty$ will be proved by a similar argument. We must show that for an arbitrary constant $M>0$, $W(\mathbf{u},x,+0)<-M$ in some neighborhood of x_0. Here, choose $r>0$ such that $W(\mathbf{u},x_0,r)\leq -2M$, and for this r take a δ-neighborhood of x_0 such that $W(\mathbf{u},x,r)\leq W(\mathbf{u},x_0,r)+M\leq -M$. □

Proof of sufficiency part of Theorem 1.2. **Case** $\kappa\notin\mathbb{N}$**:** If the statement of the theorem fails, then there exists a sequence $r_j\to 0$ such that

$$\sup_{B_r}|\mathbf{u}|\leq jr^\kappa,\quad\forall\ r\geq r_j,\qquad \sup_{B_{r_j}}|\mathbf{u}|=jr_j^\kappa.$$

In particular the function $\tilde{\mathbf{u}}_j(x)=\frac{\mathbf{u}(r_jx)}{jr_j^\kappa}$ satisfies

$$\sup_{x\in B_R}|\tilde{\mathbf{u}}_j(x)|\leq R^\kappa,\qquad \text{for } 1\leq R\leq\frac{1}{r_j},$$

with equality for $R=1$, along with

$$\Delta\tilde{\mathbf{u}}_j=\frac{\Delta\mathbf{u}(r_jx)}{jr_j^{\kappa-2}}=\frac{f(\tilde{\mathbf{u}}_j)}{j^{1-q}}\longrightarrow 0 \text{ locally uniformly.}$$

From here we conclude that $\{\tilde{\mathbf{u}}_j\}$ is bounded in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n)$ and that there is a convergent subsequence, tending to a harmonic function $\mathbf{u}_0$ with growth κ, i.e.

$$\sup_{B_R}|\mathbf{u}_0|\leq R^\kappa,\qquad\text{for all } R\geq 1,\quad \sup_{B_1}|\mathbf{u}_0|=1,\quad \Delta\mathbf{u}_0=0,\tag{7}$$

and

$$\mathbf{u}_0(0)=|\nabla\mathbf{u}_0(0)|=\cdots=|D^\ell\mathbf{u}_0(0)|=0.\tag{8}$$

Obviously (7)-(8), along with the fact that $\kappa\notin\mathbb{N}$, violates Liouville's theorem and we have a contradiction in this case.

Case $\kappa\in\mathbb{N}$**:** Let $\mathbf{u}_r(x)=\frac{\mathbf{u}(rx)}{r^\kappa}$ and notice that $\Delta\mathbf{u}_r=f(\mathbf{u}_r)$. By elliptic theory (see Theorem 8.17, in [6]) it suffices to show that $\int_{B_1}|\mathbf{u}_r|^{1+q}\,dx$ is bounded. Using monotonicity formula, Theorem 2.3, we have

$$\begin{aligned}\frac{2}{1+q}\int_{B_1}|\mathbf{u}_r|^{1+q}\le\int_{B_1}2F(\mathbf{u}_r)\le W(\mathbf{u}_r,0,1)-\int_{B_1}|\nabla\mathbf{u}_r|^2+\kappa\int_{\partial B_1}|\mathbf{u}_r|^2\\ \le W(\mathbf{u},0,r)-\int_{B_1}|\nabla\mathbf{u}_r-\nabla\mathbf{p}|^2+\kappa\int_{\partial B_1}|\mathbf{u}_r-\mathbf{p}|^2\\ \le W(\mathbf{u},0,1)+\kappa\int_{\partial B_1}|\mathbf{u}_r-\mathbf{p}|^2,\end{aligned}\tag{9}$$

where each component of $\mathbf{p}=(p_1,\dots,p_m)\in\mathbb{P}_\kappa$ is an arbitrary homogeneous harmonic polynomial of order κ. We need only to show that $\int_{\partial B_1}|\mathbf{u}_r-\pi_r|^2$ is bounded for every $r\le 1$, where $\pi_r=\operatorname{argmin}_{\mathbf{p}\in\mathbb{P}_\kappa}\int_{\partial B_1}|\mathbf{u}_r-\mathbf{p}|^2$. The function π_r satisfies

$$\int_{\partial B_1}\mathbf{p}\cdot(\mathbf{u}_r-\pi_r)\,d\mathcal{H}^{n-1}=0,\qquad\text{for every }\mathbf{p}\in\mathbb{P}_\kappa.$$

Now suppose, towards a contradiction, that there is a sequence $r_k\to 0$, such that

$$M_k=\Big(\int_{\partial B_1}|\mathbf{u}_{r_k}-\pi_{r_k}|^2\,d\mathcal{H}^{n-1}\Big)^{1/2}\longrightarrow\infty.$$

For $\mathbf{w}_k=\frac{\mathbf{u}_{r_k}-\pi_{r_k}}{M_k}$, we have $\|\mathbf{w}_k\|_{\mathcal{L}^2(\partial B_1)}=1$ and $\Delta\mathbf{w}_k=\frac{f(\mathbf{u}_{r_k})}{M_k}$,

$$\int_{B_1}|\Delta\mathbf{w}_k|^{(1+q)/q}\le\frac{1}{M_k^{(1+q)/q}}\int_{B_1}|\mathbf{u}_{r_k}|^{1+q}\le\frac{C}{M_k^{(1+q)/q}}\big(1+\int_{\partial B_1}|\mathbf{u}_{r_k}-\pi_{r_k}|^2\big)\to 0,$$

where in the last inequality we have used (9). Now $\{\mathbf{w}_k\}$ being bounded in $W^{2,2}(B_1)$ there is a weakly convergence subsequence with limit $\mathbf{w}_0$, satisfying $\Delta\mathbf{w}_0=0$, $\|\mathbf{w}_0\|_{\mathcal{L}^2(\partial B_1)}=1$ and

$$\int_{\partial B_1}\mathbf{p}\cdot\mathbf{w}_0=0,\qquad\text{for every }\mathbf{p}\in\mathbb{P}_\kappa.\tag{10}$$

On the other hand, we have

$$\begin{aligned}\int_{B_1}|\nabla\mathbf{w}_k|^2-\kappa\int_{\partial B_1}|\mathbf{w}_k|^2=&\frac{1}{M_k^2}\big(\int_{B_1}|\nabla\mathbf{u}_{r_k}|^2-\kappa\int_{\partial B_1}|\mathbf{u}_{r_k}|^2\big)\\ \le&\frac{1}{M_k^2}\big(W(\mathbf{u},0,r_k)-2\int_{B_1}F(\mathbf{u}_{r_k})dx\big)\\ \le&\frac{1}{M_k^2}W(\mathbf{u},0,1)\longrightarrow 0.\end{aligned}$$

Therefore, we obtain

$$\int_{B_1}|\nabla\mathbf{w}_0|^2-\kappa\int_{\partial B_1}|\mathbf{w}_0|^2\le 0.\tag{11}$$

On the other hand by Lemma 4.1 in [11], each component w_0^i of $\mathbf{w}_0$ must satisfy

$$\kappa\int_{\partial B_1}(w_0^i)^2\le\int_{B_1}|\nabla w_0^i|^2.$$

Summing over i and comparing with (11), this along with

$$w_0^i(0)=|\nabla w_0^i(0)|=\cdots=|D^\ell w_0^i(0)|=0$$

implies that w_0^i is a homogeneous harmonic polynomial of order κ. But (10) implies that $\mathbf{w}_0=0$ on ∂B_1 which contradicts $\|\mathbf{w}_0\|_{\mathcal{L}^2(\partial B_1)}=1$. □

Remark 2.4. *The first part of the proof, case $\kappa \notin \mathbb{N}$, works when the equation is relaxed to $|\Delta \mathbf{u}| \le c_0 |\mathbf{u}|^q$.*

3. The Epiperimetric Inequality

This section is devoted to provide the main tool of our approach, the epiperimetric inequality. Firstly, let us define the boundary adjusted energy

$$M(\mathbf{v}) := \int_{B_1} \big(|\nabla \mathbf{v}|^2 + 2F(\mathbf{v})\big)\,dx - \kappa \int_{\partial B_1} |\mathbf{v}|^2 d\mathcal{H}^{n-1}.$$

Theorem 3.1 (The epiperimetric inequlaity)**.** *There exist $\varepsilon \in (0,1)$ and $\delta > 0$ such that if $\mathbf{c} \in W^{1,2}(B_1;\mathbb{R}^m)$ is a homogeneous function of degree κ and $\|\mathbf{c} - \mathbf{h}\|_{W^{1,2}(B_1;\mathbb{R}^m)} \le \delta$ for some $\mathbf{h} \in \mathbb{H}$, then there exists a function $\mathbf{v} \in W^{1,2}(B_1;\mathbb{R}^m)$ such that $\mathbf{v} = \mathbf{c}$ on ∂B_1 and $M(\mathbf{v}) \le (1-\varepsilon) M(\mathbf{c}) + \varepsilon M(\mathbf{h})$.*

Proof. Suppose toward a contradiction that there are sequences $\varepsilon_i \to 0$, $\delta_i \to 0$, $\mathbf{c}_i \in W^{1,2}(B_1;\mathbb{R}^m)$ and $\mathbf{h}_i \in \mathbb{H}$ such that $\mathbf{c}_i$ is homogeneous of degree κ and satisfies

$$\|\mathbf{c}_i - \mathbf{h}_i\|_{W^{1,2}(B_1;\mathbb{R}^m)} = \inf_{\mathbf{h}\in\mathbb{H}} \|\mathbf{c}_i - \mathbf{h}\| = \delta_i,$$

and

$$M(\mathbf{v}) > (1-\varepsilon_i) M(\mathbf{c}_i) + \varepsilon_i M(\mathbf{h}_i), \qquad \text{for all } \mathbf{v} \in \mathbf{c}_i + W_0^{1,2}(B_1;\mathbb{R}^m). \tag{12}$$

Rotating in $\mathbb{R}^n$ and in $\mathbb{R}^m$ if necessary, we may assume that

$$\mathbf{h}_i(x) = \alpha (x_n^+)^\kappa \mathbf{e}_1 =: \mathbf{h}(x),$$

where $\mathbf{e}_1 = (1, 0, \cdots, 0) \in \mathbb{R}^m$. Notice that the energy M takes a constant value on $\mathbb{H}$, and that subtracting $M(\mathbf{h})$ from the inequality (12), we obtain

$$(1-\varepsilon_i)(M(\mathbf{c}_i) - M(\mathbf{h})) < M(\mathbf{v}) - M(\mathbf{h}), \qquad \text{for all } \mathbf{v} \in \mathbf{c}_i + W_0^{1,2}(B_1;\mathbb{R}^m). \tag{13}$$

Now observe that for all $\phi = (\phi_1, \cdots, \phi_m) \in W^{1,2}(B_1;\mathbb{R}^m)$,

$$\begin{aligned}
\delta M(\mathbf{h})(\phi) :=& 2\int_{B_1} \nabla \mathbf{h} : \nabla \phi + |\mathbf{h}|^{q-1}\mathbf{h} \cdot \phi\,dx - 2\kappa \int_{\partial B_1} \mathbf{h} \cdot \phi\, d\mathcal{H}^{n-1} \\
=& 2\int_{B_1} \big(-\Delta \mathbf{h} + f(\mathbf{h})\big) \cdot \phi\,dx + 2\int_{\partial B_1} \big(\nabla \mathbf{h} \cdot x - \kappa \mathbf{h}\big) \cdot \phi\, d\mathcal{H}^{n-1} = 0.
\end{aligned}$$

Thus we can subtract $(1-\varepsilon_i)\delta M(\mathbf{h})(\mathbf{c}_i - \mathbf{h})$ from the left hand side and $\delta M(\mathbf{h})(\mathbf{v} - \mathbf{h})$ from the right hand side of (13) to obtain

$$\begin{aligned}
(1-\varepsilon_i)\Big(&\int_{B_1} |\nabla(\mathbf{c}_i - \mathbf{h})|^2\,dx - \kappa \int_{\partial B_1} |\mathbf{c}_i - \mathbf{h}|^2\, d\mathcal{H}^{n-1} \\
&+ 2\int_{B_1} F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{c}_i - \mathbf{h})\,dx\Big) \\
&< \int_{B_1} |\nabla(\mathbf{v} - \mathbf{h})|^2\,dx - \kappa \int_{\partial B_1} |\mathbf{v} - \mathbf{h}|^2\, d\mathcal{H}^{n-1} \\
&+ 2\int_{B_1} F(\mathbf{v}) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{v} - \mathbf{h})\,dx.
\end{aligned} \tag{14}$$

Define now the normalized functions $\mathbf{w}_i := (\mathbf{c}_i - \mathbf{h})/\delta_i$, which along a subsequence converge weakly in $W^{1,2}(B_1;\mathbb{R}^m)$ to a function $\mathbf{w}$. The proof proceeds then in the following four steps:

Step 1. $\mathbf{w} = 0$ in $B_1 \cap \{x_n < 0\}$.
Step 2. $\mathbf{w}$ solves the equation $\Delta \mathbf{w} = f_{\mathbf{u}}(\mathbf{h})(\mathbf{w})$ in $B_1 \cap \{x_n > 0\}$.
Step 3. $\mathbf{w} \equiv 0$.
Step 4. $\mathbf{w}_i \to 0$ strongly in $W^{1,2}(B_1;\mathbb{R}^m)$ as the subsequence $i \to \infty$.

Since $\|\mathbf{w}_i\|_{W^{1,2}(B_1;\mathbb{R}^m)} = 1$, Step 3 and Step 4 imply a contradiction proving the theorem.

Step 1. We insert $\mathbf{v} := (1-\eta)\mathbf{c}_i + \eta\mathbf{h}$ in (14) where $\eta \in W_0^{1,2}(B_1)$ is radially symmetric and satisfies $0 \le \eta \le 1$, and obtain

$$\begin{aligned}(1-\varepsilon_i)\int_{B_1} & F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h})\cdot(\mathbf{c}_i - \mathbf{h})\,dx \\ &< C\delta_i^2 + \int_{B_1} F(\mathbf{v}) - F(\mathbf{h}) - f(\mathbf{h})\cdot(\mathbf{v}-\mathbf{h})\,dx \\ &\le C\delta_i^2 + \int_{B_1}(1-\eta)\big(F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h})\cdot(\mathbf{c}_i-\mathbf{h})\big)\,dx,\end{aligned}$$

where convexity of F is used in the last inequality. From this we obtain

$$\int_{B_1}(\eta-\varepsilon_i)\big(F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h})\cdot(\mathbf{c}_i-\mathbf{h})\big)\,dx < C\delta_i^2. \tag{15}$$

We first integrate on $B_1^+ = B_1 \cap \{x_n > 0\}$ where $|\mathbf{h}| > 0$, to arrive at

$$\begin{aligned}\int_{B_1^+} & (\eta-\varepsilon_i)\big(F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h})\cdot(\mathbf{c}_i-\mathbf{h})\big)dx \\ &= \int_{B_1^+}\int_0^1\int_0^t (\eta-\varepsilon_i)\big(f_{\mathbf{u}}(\mathbf{h}+s(\mathbf{c}_i-\mathbf{h}))(\mathbf{c}_i-\mathbf{h})\cdot(\mathbf{c}_i-\mathbf{h})\big)\,dsdtdx \\ &= \int_{B_1^+}\int_0^1\int_0^t (\eta-\varepsilon_i)\big(|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{c}_i-\mathbf{h}|^2 \\ &\qquad + (q-1)|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-3}((\mathbf{h}+s(\mathbf{c}_i-\mathbf{h}))\cdot(\mathbf{c}_i-\mathbf{h}))^2\big)\,dsdtdx \\ = \Big(\int_0^1 & (\eta(r)-\varepsilon_i)r^{k(q+1)+n-1}dr\Big)\int_{\partial B_1^+}\int_0^1\int_0^t \big(|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{c}_i-\mathbf{h}|^2 \\ &\quad + (q-1)|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-3}((\mathbf{h}+s(\mathbf{c}_i-\mathbf{h}))\cdot(\mathbf{c}_i-\mathbf{h}))^2\big)\,dsdtd\mathcal{H}^{n-1} \\ \ge \Big(\int_0^1 & (\eta(r)-\varepsilon_i)r^{k(q+1)+n-1}dr\Big)\int_{\partial B_1^+}\int_0^1\int_0^t q|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{c}_i-\mathbf{h}|^2\,dsdtd\mathcal{H}^{n-1} \\ &= \int_{B_1^+}\int_0^1\int_0^t q(\eta-\varepsilon_i)|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{c}_i-\mathbf{h}|^2\,dsdtdx.\end{aligned}$$

It is noteworthy that the phrase inside the parentheses in the last inequality is positive when ε_i is small enough. Now comparing the last inequality with (15) gives us the bound

$$\int_{B_1^-}\frac{(\eta-\varepsilon_i)}{\delta_i^{1-q}}F(\mathbf{w}_i)dx + \int_{B_1^+}\int_0^1\int_0^t q(\eta-\varepsilon_i)|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{w}_i|^2\,dsdtdx < C.$$

As $i \to \infty$ we conclude that

$$\int_{B_1^-} \eta F(\mathbf{w}) dx \leq 0.$$

Therefore $\mathbf{w} = 0$ in B_1^-.

Step 2. Insert $\mathbf{v} := \eta(\mathbf{h} + \delta_i \mathbf{g}) + (1-\eta)\mathbf{c}_i$ into (14) where $\eta \in C_0^\infty(B_1^+)$ with values in $[0,1]$ and $\mathbf{g} \in W^{1,2}(B_1; \mathbb{R}^m)$

$$\begin{aligned}\int_{B_1} |\nabla \mathbf{w}_i|^2 \, dx &+ \frac{2}{\delta_i^2} \int_{B_1} F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{c}_i - \mathbf{h}) \, dx \\ &< C\varepsilon_i + \int_{B_1} |\nabla((1-\eta)\mathbf{w}_i + \eta \mathbf{g})|^2 \, dx \\ &\quad + \frac{2}{\delta_i^2} \int_{B_1} F(\mathbf{v}) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{v} - \mathbf{h}) \, dx.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{B_1} (1 - (1-\eta)^2)|\nabla \mathbf{w}_i|^2 \, dx &+ \frac{2}{\delta_i^2} \int_{B_1^+} F(\mathbf{c}_i) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{c}_i - \mathbf{h}) \, dx \\ < C\varepsilon_i + \int_{B_1} &|\nabla(\eta \mathbf{g})|^2 + 2\nabla((1-\eta)\mathbf{w}_i) \cdot \nabla(\eta \mathbf{g}) \\ &+ |\nabla \eta|^2 |\mathbf{w}_i|^2 - 2(1-\eta)(\nabla \mathbf{w}_i \cdot \nabla \eta) \cdot \mathbf{w}_i \, dx \\ + \frac{2}{\delta_i^2} &\int_{B_1^+} F(\mathbf{v}) - F(\mathbf{h}) - f(\mathbf{h}) \cdot (\mathbf{v} - \mathbf{h}) \, dx,\end{aligned}$$

and then passing to the limit as $i \to \infty$,

$$\begin{aligned}\int_{B_1} |\nabla \mathbf{w}|^2 \, dx &+ \limsup_{i\to\infty} \int_{B_1^+} \int_0^1 \int_0^t 2 f_{\mathbf{u}}(\mathbf{h} + s\delta_i \mathbf{w}_i)(\mathbf{w}_i) \cdot \mathbf{w}_i \, dsdt \, dx \\ &\leq \int_{B_1} |\nabla((1-\eta)\mathbf{w} + \eta \mathbf{g})|^2 \, dx \\ &\quad + \liminf_{i\to\infty} \int_{B_1^+} \int_0^1 \int_0^t 2 f_{\mathbf{u}}(\mathbf{h} + s\delta_i((1-\eta)\mathbf{w}_i + \eta \mathbf{g})) \\ &\qquad ((1-\eta)\mathbf{w}_i + \eta \mathbf{g}) \cdot ((1-\eta)\mathbf{w}_i + \eta \mathbf{g}) \, dsdt \, dx. \qquad (16)\end{aligned}$$

On the other hand, in $B_1^+ \cap \operatorname{supp} \eta$, we have

$$\begin{aligned} f_{\mathbf{u}}(\mathbf{h} + s\delta_i \mathbf{w}_i)(\mathbf{w}_i) \cdot \mathbf{w}_i =& |\mathbf{h} + s\delta_i \mathbf{w}_i|^{q-1} |\mathbf{w}_i|^2 \\ &+ (q-1)|\mathbf{h} + s\delta_i \mathbf{w}_i|^{q-3}((\mathbf{h} + s\delta_i \mathbf{w}_i) \cdot \mathbf{w}_i)^2 \\ \longrightarrow & f_{\mathbf{u}}(\mathbf{h})(\mathbf{w}) \cdot \mathbf{w},\end{aligned}$$

where the convergence is valid due to the dominated convergence theorem. A similar convergence holds for the right hand side of (16), and hence

$$\begin{aligned}\int_{B_1} |\nabla \mathbf{w}|^2 \, dx &+ \int_{B_1^+ \cap \operatorname{supp} \eta} f_{\mathbf{u}}(\mathbf{h})(\mathbf{w}) \cdot \mathbf{w} \, dx \\ &\leq \int_{B_1} |\nabla((1-\eta)\mathbf{w} + \eta \mathbf{g})|^2 \, dx \\ &\quad + \int_{B_1^+ \cap \operatorname{supp} \eta} f_{\mathbf{u}}(\mathbf{h})((1-\eta)\mathbf{w} + \eta \mathbf{g}) \cdot ((1-\eta)\mathbf{w} + \eta \mathbf{g}) \, dx.\end{aligned}$$

Consider an open ball $B \subset B_1 \cap \{x_n > 0\}$. We may choose $\eta := 1$ in B and $\mathbf{g} := \mathbf{w}$ outside B to obtain that

$$\int_B |\nabla \mathbf{w}|^2\,dx + \int_B f_{\mathbf{u}}(\mathbf{h})(\mathbf{w}) \cdot \mathbf{w}\,dx \le \int_B |\nabla \mathbf{g}|^2\,dx + \int_B f_{\mathbf{u}}(\mathbf{h})(\mathbf{g}) \cdot \mathbf{g}\,dx,$$

for all $\mathbf{g} \in W^{1,2}(B;\mathbb{R}^m)$ coinciding with $\mathbf{w}$ on ∂B. Therefore,

$$\Delta \mathbf{w} = |\mathbf{h}|^{q-1}\mathbf{w} + (q-1)|\mathbf{h}|^{q-3}(\mathbf{h}\cdot\mathbf{w})\mathbf{h}.$$

Step 3. Let $w_j := \mathbf{w}\cdot\mathbf{e}_j$ for $1 \le j \le m$, then

$$\Delta w_j = q\kappa(\kappa-1)(x_n^+)^{-2} w_j, \qquad \text{for } j = 1,$$

and

$$\Delta w_j = \kappa(\kappa-1)(x_n^+)^{-2} w_j, \qquad \text{for } j > 1.$$

Now extend w_j to a homogeneous function of degree κ in $\{x_n > 0\}$ and define

$$\tilde{w}_j(x', x_n) := \begin{cases} w_j(x', x_n), & x_n > 0, \\ -w_j(x', -x_n), & x_n < 0, \end{cases}$$

which is a homogeneous weak solution of degree κ and satisfies

$$\Delta \tilde{w}_j = \begin{cases} q\kappa(\kappa-1)|x_n|^{-2}\tilde{w}_j, & \text{for } j = 1, \\ \kappa(\kappa-1)|x_n|^{-2}\tilde{w}_j, & \text{for } j > 1. \end{cases} \tag{17}$$

Note that as a result of Step 1, the trace of $\mathbf{w}$ vanishes on $\{x_n = 0\}$. If we consider now any multiindex $\mu \in \mathbb{Z}_+^{n-1} \times \{0\}$ and the higher order partial derivatives $\partial^\mu \tilde{w}_j =: \zeta$ then ζ satisfies again in the same equation in $\mathbb{R}^n$. Then ζ is by repeated local energy estimates contained in $W^{1,2}_{loc}(\mathbb{R}^n)$ and ζ is a homogeneous function of degree $\kappa - |\mu|_1$. From the integrability and homogeneity we infer that $\partial^\mu \tilde{w}_j \equiv 0$ for $\kappa - |\mu|_1 - 1 \le -n/2$. Thus $x' \mapsto \tilde{w}_j(x', x_n)$ is a polynomial and the homogeneity and integrability imply the existence of a polynomial p of degree $\deg p < \kappa + \frac{1}{2} - 1$, such that $w_j(x', x_n) = x_n^\kappa p(\frac{x'}{x_n})$ for $x_n > 0$. Next we take $\mu \in \mathbb{Z}_+^{n-1} \times \{0\}$ such that $|\mu|_1 = \deg p$, then $\partial^\mu w_j = \partial^\mu p x_n^{\kappa - |\mu|_1}$. Comparing with the equation (17), in the case that $\partial^\mu p \ne 0$, implies that

$$(\kappa - |\mu|_1)(\kappa - |\mu|_1 - 1) = q\kappa(\kappa - 1), \quad \text{for } j = 1,$$

$$(\kappa - |\mu|_1)(\kappa - |\mu|_1 - 1) = \kappa(\kappa - 1), \quad \text{for } j > 1,$$

and hence

$$|\mu|_1 = 1, \text{ or } 2\kappa - 2, \quad \text{for } j = 1,$$

$$|\mu|_1 = 0, \text{ or } 2\kappa - 1, \quad \text{for } j > 1.$$

On the other hand, we have $\deg p < \kappa - \frac{1}{2}$, and only the cases $|\mu|_1 = 1$ for $j = 1$ and $|\mu|_1 = 0$ for $j > 1$ are possible. For $j = 1$, we obtain that $w_1(x) = x_n^\kappa(d + \ell\cdot x'/x_n)$, whereupon the equation for w_1 yields that

$$\Delta w_1 = (\kappa-1)(\kappa-2)x_n^{\kappa-3}(dx_n + \ell\cdot x') + 2d(\kappa-1)x_n^{\kappa-2} = q\kappa(\kappa-1)x_n^{\kappa-3}(dx_n + \ell\cdot x').$$

We deduce that $d = 0$ and that $w_1(x) = x_n^{\kappa-1}\ell\cdot x'$ in $\{x_n > 0\}$. By similar argument for $j > 1$, we find that $\mathbf{w}(x) = (x_n^{\kappa-1}\ell_1\cdot x', \ell_2 x_n^\kappa, \ldots, \ell_m x_n^\kappa)$.

Recall that we have chosen $\mathbf{h}$ as the best approximation of $\mathbf{c}_i$ in $\mathbb{H}$. It follows that for $\mathbf{h}_\nu(x) := \alpha\max(x\cdot\nu,0)^\kappa\mathbf{e}_1$,

$$(\mathbf{w}_i,\mathbf{h}_\nu-\mathbf{h})_{W^{1,2}(B_1;\mathbb{R}^m)} \le \frac{1}{2\delta_i}\|\mathbf{h}_\nu-\mathbf{h}\|^2_{W^{1,2}(B_1;\mathbb{R}^m)}. \tag{18}$$

Now let $\nu\to\mathbf{e}_n$ so that $\frac{\nu-\mathbf{e}_n}{|\nu-\mathbf{e}_n|}$ converges to the vector ξ (where $\xi\cdot\mathbf{e}_n=0$), then

$$\begin{aligned} o(1) \ge \int_{B_1} &(\mathbf{w}_i\cdot\mathbf{e}_1)\kappa(x_n^+)^{\kappa-1}(x\cdot\xi)+\\ &\nabla(\mathbf{w}_i\cdot\mathbf{e}_1)\cdot\Big[\kappa(x_n^+)^{\kappa-1}\xi+\kappa(\kappa-1)(x_n^+)^{\kappa-2}(x\cdot\xi)\mathbf{e}_n\Big]dx. \end{aligned}$$

Choosing $\xi=(\ell_1,0)$ and passing to the limit in i, we obtain that

$$\begin{aligned} 0 \ge \int_{B_1} &\kappa(x_n^+)^{2\kappa-2}(x'\cdot\ell_1)^2+\kappa(x_n^+)^{2\kappa-2}|\ell_1|^2\\ &+\kappa(\kappa-1)^2(x_n^+)^{2\kappa-4}(x'\cdot\ell_1)^2dx. \end{aligned}$$

Hence, $\ell_1=0$, and $\mathbf{w}\cdot\mathbf{e}_1=0$.

It remains to show that $\mathbf{w}\cdot\mathbf{e}_j=0$ for $j>1$. Apply once more the relation (18) for $\mathbf{h}_t=\alpha(x_n^+)^\kappa\mathbf{e}_t$ instead of $\mathbf{h}_v$, where $\mathbf{e}_t=(\cos t)\mathbf{e}_1\pm(\sin t)\mathbf{e}_j$, and let $t\to0$. We obtain

$$(\mathbf{w}_i,\pm\alpha(x_n^+)^\kappa\mathbf{e}_j)_{W^{1,2}(B_1;\mathbb{R}^m)}\le 0.$$

Therefore,

$$\ell_j\|(x_n^+)^\kappa\|^2_{W^{1,2}(B_1)}=0,$$

and $\ell_j=0$.

Step 4. In order to show the strong convergence of $\mathbf{w}_i$ in $W^{1,2}(B_1;\mathbb{R}^m)$, choose $\mathbf{v}:=(1-\eta)\mathbf{c}_i+\eta\mathbf{h}$ as a test function in (14), where $\eta=:\max(0,\min(1,2(1-|x|)))$. Then as in Step 1, we obtain

$$\begin{aligned} \int_{B_1}|\nabla\mathbf{w}_i|^2dx+&\int_{B_1}\frac{\eta-\varepsilon_i}{\delta_i^2}\big[F(\mathbf{c}_i)-F(\mathbf{h})-f(\mathbf{h})\cdot(\mathbf{c}_i-\mathbf{h})\big]\,dx\\ &\le C\varepsilon_i+\int_{B_1}|\nabla((1-\eta)\mathbf{w}_i)|^2dx\\ &=C\varepsilon_i+\int_{B_1}(1-\eta)^2|\nabla\mathbf{w}_i|^2-2(1-\eta)(\nabla\mathbf{w}_i\cdot\nabla\eta)\cdot\mathbf{w}_i+|\nabla\eta|^2|\mathbf{w}_i|^2dx \end{aligned}$$

and

$$\begin{aligned} \int_{B_{1/2}}|\nabla\mathbf{w}_i|^2dx+&\int_{B_1^-}\eta F(\mathbf{w}_i)dx+\int_{B_1^+}\int_0^1\int_0^t q\eta|\mathbf{h}+s(\mathbf{c}_i-\mathbf{h})|^{q-1}|\mathbf{w}_i|^2\,dsdtdx\\ &\le C\varepsilon_i+\int_{B_1}|\nabla\eta|^2|\mathbf{w}_i|^2-2(1-\eta)(\nabla\mathbf{w}_i\cdot\nabla\eta)\cdot\mathbf{w}_idx. \end{aligned}$$

At this point, notice that second and third integral in the last relation are positive and use the homogeneity of $\mathbf{w}_i$ to obtain

$$\begin{aligned} \int_{B_1}|\nabla\mathbf{w}_i|^2\,dx&=2^{n+2\kappa-2}\int_{B_{1/2}}|\nabla\mathbf{w}_i|^2\,dx\\ &\le 2^{n+2\kappa-2}\Big(C\varepsilon_i+\int_{B_1}|\nabla\eta|^2|\mathbf{w}_i|^2-2(1-\eta)(\nabla\mathbf{w}_i\cdot\nabla\eta)\cdot\mathbf{w}_idx\Big)\to0. \end{aligned}$$

□

4. Regularity of Free Boundary at Regular Points

In this section, we will study the regularity of the free boundary near a one-phase point at which at least one blow-up limit coincides with a half-plane solution.

4.1. Nondegeneracy.

Proposition 4.1. *Let* $\mathbf{u}$ *be a solution of* (3) *with* $0 < q < 1$. *Then there is a positive constant* $c = c(q, n)$ *such that if* $x_0 \in \overline{\{|\mathbf{u}| > 0\}}$ *and* $B_r(x_0) \subset B_1$, *then*

$$\sup_{B_r(x_0)} |\mathbf{u}| \geq cr^\kappa.$$

Proof. Let $v(x) := |\mathbf{u}(x)|^{1-q}$. Then

$$\Delta v = (1-q) + (1-q)\frac{|\nabla \mathbf{u}|^2}{v^{\kappa-1}} - \frac{1+q}{1-q}\frac{|\nabla v|^2}{v}.$$

For any $y \in \{|\mathbf{u}| > 0\}$ (close to x_0), set $w(x) = c|x-y|^2$ for small constant $c > 0$ to be specified later. Then $h = v - w$ satisfies in $\{|\mathbf{u}| > 0\}$

$$\mathcal{L}h := \Delta h + \frac{1+q}{1-q}\big(\frac{\nabla(v+w)}{v}\cdot\nabla h - \frac{4c}{v}h\big) = (1-q) - 4c(\frac{n}{2} + \frac{1+q}{1-q}) + (1-q)\frac{|\nabla \mathbf{u}|^2}{v^{\kappa-1}} \geq 0,$$

provided that c is small enough. In particular h cannot attain a local maximum in $B_r(y) \cap \{|\mathbf{u}| > 0\}$. On the other hand $h < 0$ on $\partial\{|\mathbf{u}| > 0\}$ and hence the positive maximum of h is attained on $\partial B_r(y)$, and we conclude that

$$\sup_{\partial B_r(y)\cap\{|\mathbf{u}|>0\}} (v - w) \geq v(y) > 0,$$

which amounts to

$$\sup_{\partial B_r(y)\cap\{|\mathbf{u}|>0\}} v \geq cr^2.$$

Letting $y \to x_0$, we arrive at the statement of the lemma. □

4.2. Energy decay.

Theorem 4.2 (Energy decay)**.** *Let* $x_0 \in B_1 \cap \partial\{|\mathbf{u}| > 0\}$, *and suppose that the epiperimetric inequality holds with* $\varepsilon \in (0,1)$ *for each*

$$\mathbf{c}_r(x) := |x|^\kappa \mathbf{u}_r(\frac{x}{|x|}) = \frac{|x|^\kappa}{r^\kappa}\mathbf{u}(x_0 + \frac{r}{|x|}x)$$

and for all $r \leq r_0 < 1$. *Finally let* $\mathbf{u}_0$ *denote an arbitrary blow-up limit of* $\mathbf{u}$ *at* x_0 *and* $\Lambda = (n + 2\kappa - 2)\varepsilon/(1-\varepsilon)$. *Then*

$$|W(\mathbf{u}, x_0, r) - W(\mathbf{u}, x_0, 0+)| \leq |W(\mathbf{u}, x_0, r_0) - W(\mathbf{u}, x_0, 0+)| \left(\frac{r}{r_0}\right)^\Lambda,$$

for $r \in (0, r_0)$, *and there exists a constant* C *depending only on* n *and* ε *such that*

$$\int_{\partial B_1} |\mathbf{u}_r(x) - \mathbf{u}_0(x)| d\mathcal{H}^{n-1} \leq C|W(\mathbf{u}, x_0, r_0) - W(\mathbf{u}, x_0, 0+)|^{1/2}(\frac{r}{r_0})^{\Lambda/2}.$$

Proof. We define

$$e(r) := W(\mathbf{u}, x_0, r) - W(\mathbf{u}, x_0, 0+) = r^{-n-2\kappa+2}\int_{B_r(x_0)} \big(|\nabla \mathbf{u}|^2 + 2F(\mathbf{u})\big)\,dx - \kappa r^{-n-2\kappa+1}\int_{\partial B_r(x_0)} |\mathbf{u}|^2\,d\mathcal{H}^{n-1} - W(\mathbf{u}, x_0, 0+)$$

and calculate

$$\begin{aligned}
e'(r) =& -\frac{n+2\kappa-2}{r}\big(e(r)+W(\mathbf{u},x_0,0+)\big)+\kappa r^{-n-2\kappa}\int_{\partial B_r(x_0)}|\mathbf{u}|^2\,d\mathcal{H}^{n-1}\\
&+r^{-n-2\kappa+2}\int_{\partial B_r(x_0)}\big(|\nabla\mathbf{u}|^2+2F(\mathbf{u})\big)d\mathcal{H}^{n-1}\\
&-2\kappa r^{-n-2\kappa+1}\int_{\partial B_r(x_0)}(\nabla\mathbf{u}\cdot\nu)\cdot\mathbf{u}d\mathcal{H}^{n-1}-\kappa(n-1)r^{-n-2\kappa}\int_{\partial B_r(x_0)}|\mathbf{u}|^2\,d\mathcal{H}^{n-1}\\
=& -\frac{n+2\kappa-2}{r}\big(e(r)+W(\mathbf{u},x_0,0+)\big)-\frac{\kappa}{r}(n-2)\int_{\partial B_1}|\mathbf{u}_r|^2\,d\mathcal{H}^{n-1}\\
&+\frac{1}{r}\int_{\partial B_1}\big(|\nabla\mathbf{u}_r|^2+2F(\mathbf{u}_r)\big)d\mathcal{H}^{n-1}-\frac{2\kappa}{r}\int_{\partial B_1}(\nabla\mathbf{u}_r\cdot\nu)\cdot\mathbf{u}_r d\mathcal{H}^{n-1}\\
\geq& -\frac{n+2\kappa-2}{r}\big(e(r)+W(\mathbf{u},x_0,0+)\big)\\
&+\frac{1}{r}\int_{\partial B_1}|\nabla_\theta\mathbf{u}_r|^2-(\kappa(n-2)+\kappa^2)|\mathbf{u}_r|^2+2F(\mathbf{u}_r)d\mathcal{H}^{n-1}\\
=& -\frac{n+2\kappa-2}{r}\big(e(r)+W(\mathbf{u},x_0,0+)\big)\\
&+\frac{1}{r}\int_{\partial B_1}|\nabla_\theta\mathbf{c}_r|^2-(\kappa(n-2)+\kappa^2)|\mathbf{c}_r|^2+2F(\mathbf{c}_r)d\mathcal{H}^{n-1}\\
=& -\frac{n+2\kappa-2}{r}\big(e(r)+W(\mathbf{u},x_0,0+)-M(\mathbf{c}_r)\big).
\end{aligned}$$

At this point, we employ the minimality of $\mathbf{u}$ as well as the assumption that the epiperimetric inequality $M(\mathbf{v})\leq(1-\varepsilon)M(\mathbf{c}_r)+\varepsilon W(\mathbf{u},x_0,0+)$ holds for some $\mathbf{v}\in W^{1,2}(B_1;\mathbb{R}^m)$ with $\mathbf{c}_r$-boundary values and we obtain for $r\in(0,r_0)$ the estimate

$$\begin{aligned}
e'(r)\geq&\frac{n+2\kappa-2}{r}\Big(\frac{1}{1-\varepsilon}(M(\mathbf{u}_r)-W(\mathbf{u},x_0,0+))-e(r)\Big)\\
=&\frac{n+2\kappa-2}{r}\Big(\frac{1}{1-\varepsilon}-1\Big)e(r)=\frac{n+2\kappa-2}{r}\frac{\varepsilon}{1-\varepsilon}e(r).
\end{aligned}$$

By the monotonicity formula Proposition 2.3, $e(r)\geq 0$, and we conclude in the non-trivial case $e>0$ that in (r_1,r_0)

$$e(r)\leq e(r_0)\Big(\frac{r}{r_0}\Big)^{\Lambda}\quad\text{for } r\in(r_1,r_0),$$

which proves the first statement.

Now using once more the monotonicity formula, Proposition 2.3, we get for $0 < \rho < \sigma \le r_0$ an estimate of the form

$$\begin{aligned}\int_{\partial B_1} |\mathbf{u}_\sigma(x) - \mathbf{u}_\rho(x)|\, d\mathcal{H}^{n-1} &\le \int_{\partial B_1} \int_\rho^\sigma |\partial_r \mathbf{u}_r| dr\, d\mathcal{H}^{n-1} \\ &= \int_\rho^\sigma r^{-1} \int_{\partial B_1} \big|\nabla \mathbf{u}_r \cdot x - \kappa \mathbf{u}_r\big|\, d\mathcal{H}^{n-1} dr \\ &\le \int_\rho^\sigma r^{-1/2} \sqrt{\frac{n\omega_n}{2}} \Big(\int_{\partial B_1} \frac{2}{r} \big|\nabla \mathbf{u}_r \cdot x - \kappa \mathbf{u}_r\big|^2 d\mathcal{H}^{n-1} \Big)^{1/2} dr \\ &= \sqrt{\frac{n\omega_n}{2}} \int_\rho^\sigma r^{-1/2} \sqrt{e'(r)} dr \\ &\le \sqrt{\frac{n\omega_n}{2}} (\log(\sigma) - \log(\rho))^{1/2} (e(\sigma) - e(\rho))^{1/2}.\end{aligned}$$

Considering now $0 < 2\rho < 2r \le r_0$ and intervals $[2^{-k-1}, 2^{-k}) \ni \rho$ and $[2^{-\ell-1}, 2^{-\ell}) \ni r$ the already proved part of the theorem yields that

$$\begin{aligned}\int_{\partial B_1} |\mathbf{u}_r(x) - \mathbf{u}_\rho(x)|\, d\mathcal{H}^{n-1} &\le C_1(n) \sum_{i=\ell}^{k} (\log(2^{-i}) - \log(2^{-i-1}))^{1/2} (e(2^{-i}) - e(2^{-i-1}))^{1/2} \\ &\le C_2(n) \sum_{i=\ell}^{k} (e(2^{-i}) - e(2^{-i-1}))^{1/2} \\ &\le C_2(n)(e(r_0))^{1/2} \sum_{i=\ell}^{k} (r_0 2^i)^{-\Lambda/2} \\ &\le C_2(n)(e(r_0))^{1/2} r_0^{-\Lambda/2} \sum_{i=\ell}^{\infty} 2^{-i\Lambda/2} \\ &\le C_3(n, \kappa, \varepsilon)(e(r_0))^{1/2} (r_0 2^\ell)^{-\Lambda/2} \\ &\le C_3(n, \kappa, \varepsilon)(e(r_0))^{1/2} (\frac{2r}{r_0})^{\Lambda/2}.\end{aligned}$$

□

4.3. **Uniqueness of blow-up.** In this subsection, we will show the uniqueness of blow-up and estimate the rate of convergence of the scaled solution to its blow-up in Theorem 4.7.

First we need to prove some preliminaries.

Lemma 4.3. *The half-plane solutions of the system* (3) *are isolated (in the topology of $W^{1,2}(B_1(0); \mathbb{R}^m)$) within the class of homogeneous solutions of degree κ.*

Proof. We suppose towards a contradiction that this does not hold. Then there exists a sequence of homogeneous solutions of degree κ, say $\mathbf{u}_n$, such that

$$0 < \inf_{\mathbf{h} \in \mathbb{H}} \|\mathbf{u}_n - \mathbf{h}\|_{W^{1,2}(B_1(0);\mathbb{R}^m)} = \|\mathbf{u}_n - \hat{\mathbf{h}}\|_{W^{1,2}(B_1(0);\mathbb{R}^m)} =: \delta_n \to 0, \quad \text{as } n \to \infty,$$

where $\hat{\mathbf{h}} = \alpha (x_n^+)^\kappa \mathbf{e}_1$. When passing to a subsequence, $(\mathbf{u}_n - \hat{\mathbf{h}})/\delta_n =: \mathbf{w}_n \rightharpoonup \mathbf{w}$ weakly in $W^{1,2}(B_1(0); \mathbb{R}^m)$, the limit $\mathbf{w}$ is still a homogeneous function of degree

κ. Furthermore, for $\phi \in C_0^\infty(B_1;\mathbb{R}^m)$ we have

$$\begin{aligned}-\int_{B_1} \nabla \mathbf{w}_n \cdot \nabla \phi \, dx =& \frac{1}{\delta_n}\int_{B_1} (f(\mathbf{u}_n) - f(\hat{\mathbf{h}}))\cdot \phi \, dx \\ =& \frac{1}{\delta_n}\int_{B_1}\int_0^1 \frac{d}{dt} f(\hat{\mathbf{h}} + t(\mathbf{u}_n - \hat{\mathbf{h}}))\cdot \phi \, dt dx \\ =& \int_{B_1}\int_0^1 f_{\mathbf{u}}(\hat{\mathbf{h}} + t\delta_n \mathbf{w}_n)(\mathbf{w}_n)\cdot \phi \, dt dx.\end{aligned}$$

If $\operatorname{supp}\phi \subset B_1^-$, let $n\to\infty$ we conclude that

$$\begin{aligned}\int_{B_1^-} f_{\mathbf{u}}(\mathbf{w})(\mathbf{w})\cdot\phi \, dx =& \lim_{n\to\infty}\int_{B_1^-} f_{\mathbf{u}}(\mathbf{w}_n)(\mathbf{w}_n)\cdot \phi\, dx \\ =& -\lim_{n\to\infty}\int_{B_1^-} q\delta_n^{1-q}\nabla \mathbf{w}_n \cdot \nabla\phi \, dx = 0.\end{aligned}$$

Then $\mathbf{w}\equiv 0$ in $B_1(0)\cap\{x_n<0\}$. Now for every $\operatorname{supp}\phi\subset B_1^+$,

$$-\int_{B_1^+} \nabla \mathbf{w}\cdot\nabla\phi \, dx = \int_{B_1^+} f_{\mathbf{u}}(\hat{\mathbf{h}})(\mathbf{w})\cdot\phi\, dx.$$

Thus $\Delta \mathbf{w} = f_{\mathbf{u}}(\hat{\mathbf{h}})(\mathbf{w})$ in $B_1(0)\cap\{x_n>0\}$.

On the other hand, we know that $\hat{\mathbf{h}}$ is the best approximation to $\mathbf{u}_n$ among all half-plane solutions. But then it follows exactly as in Step 3 of the proof of the epiperimetric inequality Theorem 3.1 that $\mathbf{w}\equiv 0$. In order to obtain a contradiction to the assumption $\delta_n > 0$ by which $\|\mathbf{w}_n\|_{W^{1,2}(B_1(0);\mathbb{R}^m)} = 1$, it is therefore sufficient to show the strong convergence of $\nabla \mathbf{w}_n$ to $\nabla \mathbf{w}$ in $\mathcal{L}^2(B_1(0);\mathbb{R}^{mn})$ as the subsequence $n\to\infty$. But by compact imbedding on the boundary

$$\begin{aligned}\int_{B_1} |\nabla \mathbf{w}_n|^2 =& \int_{\partial B_1} \mathbf{w}_n\cdot\nabla\mathbf{w}_n \nu \, d\mathcal{H}^{n-1} - \int_{B_1} \mathbf{w}_n\cdot\Delta\mathbf{w}_n \\ =& \kappa\int_{\partial B_1} |\mathbf{w}_n|^2 \, d\mathcal{H}^{n-1} - \frac{1}{\delta_n^2}\int_{B_1}(\mathbf{u}_n - \hat{\mathbf{h}})\cdot(f(\mathbf{u}_n) - f(\hat{\mathbf{h}}))\, dx \\ \le& \kappa\int_{\partial B_1} |\mathbf{w}_n|^2 \, d\mathcal{H}^{n-1} \to 0,\end{aligned}$$

as the subsequence $n\to\infty$. □

Remark 4.4. *Theorem 4.2 proves the uniqueness of blow-ups provided $\mathbf{u}_r$ remains in a δ-neighborhood of $\mathbb{H}$, where δ is the constant introduced in the epiperimetric inequality. Lemma 4.3, however, provides this condition.*

Proposition 4.5. *Let $\mathbf{u}\not\equiv 0$ be a homogeneous solution of degree κ satisfying $\{|\mathbf{u}|=0\}^\circ \neq \varnothing$. Then $M(\mathbf{u}) \ge \alpha_n/2$, and equality implies that $\mathbf{u}$ is a half-plane solution; here $\alpha_n = 2M(\mathbf{h})$ for every $\mathbf{h}\in\mathbb{H}$.*

Proof. The proof is by induction on n, the dimension of the domain. In one space dimension the statement is an immediate consequence of the homogeneity. We assume that it holds for every solution in dimension $\le n-1$ and that it is violated by a homogeneous solution $\mathbf{u}$ of degree κ in dimension n, that $\{|\mathbf{u}|=0\}$ contains the ball B and that $\mathbf{e}_n \in \partial B\cap\partial\{|\mathbf{u}|>0\}$. The homogeneity of $\mathbf{u}$ implies that

$$W(\mathbf{u},\mathbf{e}_n,0+) = \lim_{r\to 0^+} W(\mathbf{u},\mathbf{e}_n,r) = \lim_{r\to 0^+} W(\mathbf{u},\frac{\mathbf{e}_n}{m},\frac{r}{m}) = W(\mathbf{u},\frac{\mathbf{e}_n}{m},0+),$$

and by the upper semicontinuity of the function $x \mapsto W(\mathbf{u}, x, 0+)$,

$$W(\mathbf{u}, \mathbf{e}_n, 0+) = \limsup_{m\to\infty} W(\mathbf{u}, \frac{\mathbf{e}_n}{m}, 0+) \le W(\mathbf{u}, 0, 0+) \le W(\mathbf{u}, 0, 1) = M(\mathbf{u}) < \frac{\alpha_n}{2}.$$

Thus every blow-up limit $\mathbf{u}_0$ of $\mathbf{u}$ at the point $\mathbf{e}_n$ satisfies the inequality $M(\mathbf{u}_0) < \alpha_n/2$. (Note that by the nondegeneracy property $\mathbf{u}_0 \not\equiv 0$.) Now the homogeneity of $\mathbf{u}$ tells us that $\mathbf{u}_0$ must be constant in the direction of the vector $\mathbf{e}_n$ and that again $\{|\mathbf{u}_0| = 0\}^\circ \neq \varnothing$, so $\tilde{\mathbf{u}} := \mathbf{u}_0|_{\mathbb{R}^{n-1}}$ is a homogeneous solution of degree κ satisfying $\{|\tilde{\mathbf{u}}| = 0\}^\circ \neq \varnothing$, and

$$\begin{aligned}
\frac{\alpha_n}{2} > &\int_{B_1} |\nabla \mathbf{u}_0|^2 + 2F(\mathbf{u}_0)\,dx - \kappa \int_{\partial B_1} |\mathbf{u}_0|^2 \,d\mathcal{H}^{n-1} = \frac{1-q}{1+q}\int_{B_1} |\mathbf{u}_0|^{1+q}\,dx \\
=&\frac{2(1-q)}{1+q}\int_{\{|x'|<1\}}\int_0^{\sqrt{1-|x'|^2}} |\tilde{\mathbf{u}}(x')|^{1+q}\,dx_n dx' \\
=&\frac{2(1-q)}{1+q}\int_0^1 \sqrt{1-r^2}r^{n-2+\kappa(1+q)}dr \int_{\partial B_1'} |\tilde{\mathbf{u}}(x')|^{1+q}\,d\mathcal{H}^{n-2} \\
=&2M(\tilde{\mathbf{u}})\int_0^1 \sqrt{1-r^2}r^{n-2+\kappa(1+q)}dr \\
\ge&\alpha_{n-1}\int_0^1 \sqrt{1-r^2}r^{n-2+\kappa(1+q)}dr = \frac{\alpha_n}{2},
\end{aligned}$$

which contradicts the induction hypothesis. (Notice that the last equality is obtained by the same calculation for $\mathbf{h}(x) = \alpha(x_1^+)^\kappa \mathbf{e} \in \mathbb{H}$ instead of $\mathbf{u}_0$.)

Finally, we assume inductively that the second part of the statement holds for every dimension $\le n-1$ and consider the case of a homogeneous solution $\mathbf{u}$ of degree κ in dimension n satisfying $M(\mathbf{u}) = \alpha_n/2$, $B \subset \{|\mathbf{u}| = 0\}$ and $\mathbf{e}_n \in \partial B \cap \partial\{|\mathbf{u}| > 0\}$. As in the first part of the proof we obtain that every blow-up limit $\mathbf{u}_0$ of $\mathbf{u}$ at the point $\mathbf{e}_n$ satisfies the inequlity $M(\mathbf{u}_0) \le \alpha_n/2$, that $\mathbf{u}_0$ is constant in the direction of $\mathbf{e}_n$ and that $\{|\mathbf{u}_0| = 0\}^\circ \neq \varnothing$. Defining again $\tilde{\mathbf{u}} := \mathbf{u}_0|_{\mathbb{R}^{n-1}}$, which is a homogeneous solution of degree κ satisfying $\{|\tilde{\mathbf{u}}| = 0\}^\circ \neq \varnothing$, the calculation in the first part of the proof yields that $M(\tilde{\mathbf{u}}) \le \alpha_{n-1}/2$. Thus $\tilde{\mathbf{u}}$ must be a half-plane solution by the induction hypothesis, and so must $\mathbf{u}_0$. Therefore, for $0 < r_m \to 0$, every blow-up limit of $\mathbf{u}$ at the point $r_m\mathbf{e}_n$ must be a half-plane solution. Assuming that $\mathbf{u} \notin \mathbb{H}$, we find by a continuity argument for an arbitrary $\theta \in (0,1)$ a sequence $\rho_m \to 0$ such that

$$\mathit{dist}(\rho_m^{-\kappa}\mathbf{u}(r_m\mathbf{e}_n + \rho_m\cdot), \mathbb{H}) = \theta\,\mathit{dist}(\mathbf{u}, \mathbb{H}) > 0,$$

where the distance is measured in the $W^{1,2}(B_1(0); \mathbb{R}^m)$-norm. On the other hand, it follows that $\mathbf{u}(r_m\mathbf{e}_n + \rho_m\cdot)/\rho_m^\kappa$ converges in $W^{1,2}(B_1(0); \mathbb{R}^m)$ to a homogeneous solution $\mathbf{u}^*$ of degree κ along a subsequence $m \to \infty$. Although the boundedness of $\mathbf{u}(r_m\mathbf{e}_n + \rho_m\cdot)/\rho_m^\kappa$ implies the weak convergence in $W^{1,2}(B_1(0); \mathbb{R}^m)$, the compact embedding on the boundary proves the strong convergence as at the end of the proof of Lemma 4.3. The conclusion is that $\mathit{dist}(\mathbf{u}^*, \mathbb{H}) = \theta\,\mathit{dist}(\mathbf{u}, \mathbb{H}) > 0$ which for small θ contradicts the isolation property Lemma 4.3. □

Remark 4.6. *Proposition 4.5 and Lemma 4.3 show that the infimum energy of all κ-homogeneous solutions outside of $\mathbb{H}$ is strictly greater than $\alpha_n/2$. From this fact we infer that the set of regular free boundary points $\mathcal{R}_{\mathbf{u}}$ is open relative to $\Gamma(\mathbf{u})$.*

Theorem 4.7. *Let C_h be a compact subset of $\mathcal{R}_{\mathbf{u}}$. Assume that $\mathbf{u}_0(x) = \alpha \max(x\cdot \nu(x_0), 0)^\kappa \mathbf{e}(x_0)$ is the blow-up limit of $\mathbf{u}$ at x_0 which is a half-plane solution, for some $\nu(x_0) \in \partial B_1(0) \subset \mathbb{R}^n$ and $\mathbf{e}(x_0) \in \partial B_1(0) \subset \mathbb{R}^m$. Then there exists $r_0 > 0$ and positive constant C, such that*

$$\int_{\partial B_1} \Big| \frac{\mathbf{u}(x_0 + rx)}{r^\kappa} - \alpha \max(x\cdot \nu(x_0), 0)^\kappa \mathbf{e}(x_0) \Big| \, d\mathcal{H}^{n-1} \le C r^{\Lambda/2},$$

for every $x_0 \in C_h$ and every $r \le r_0$. Here, Λ is the exponent defined in Theorem 4.2.

Proof. In view of Theorem 3.1 and Theorem 4.2, it is sufficient to show that

$$dist(\frac{\mathbf{u}(x_0 + r\,\cdot)}{r^\kappa}, \mathbb{H}) \le \delta,$$

for every $x_0 \in C_h$ and $r \le r_0$, where the distance is measured in the $W^{1,2}(B_1(0);\mathbb{R}^m)$-norm. Assume now towards a contradiction that $dist(\frac{\mathbf{u}(x_i+\rho_i\,\cdot)}{\rho_i^\kappa}, \mathbb{H}) \ge \delta > 0$ for some $x_i \in C_h$ and $\rho_i \to 0$. By a continuity argument, for each $\theta \in (0, 1)$ there is a sequence $\tilde\rho_i < \rho_i$ such that $dist(\frac{\mathbf{u}(x_i+\tilde\rho_i\,\cdot)}{\tilde\rho_i^\kappa}, \mathbb{H}) = \theta\delta$. Then $\mathbf{u}_i := \frac{\mathbf{u}(x_i+\tilde\rho_i\,\cdot)}{\tilde\rho_i^\kappa}$ is bounded in $W^{1,2}(B_1(0);\mathbb{R}^m)$, and passing to a limit with respect to a subsequence we obtain a solution $\mathbf{u}_h$ satisfying $dist(\mathbf{u}_h, \mathbb{H}) = \theta\delta$. Moreover,

$$W(\mathbf{u}_h, 0, r) = \lim_{i\to\infty} W(\mathbf{u}_i, 0, r) = \lim_{i\to\infty} W(\mathbf{u}, x_i, r\tilde\rho_i) = \alpha_n/2.$$

Thus $\mathbf{u}_h$ is a κ-homogeneous solution by Proposition 2.3 and so for small θ contradicts the isolation property in Lemma 4.3. □

4.4. **Proof of Theorem 1.4.** Let us consider $x_0 \in \mathcal{R}_{\mathbf{u}}$. By Theorem 4.7 there exists $\delta_0 > 0$ such that $B_{2\delta_0}(x_0) \subset B_1$, $B_{2\delta_0}(x_0) \cap \partial\{|\mathbf{u}| > 0\} = B_{2\delta_0}(x_0) \cap \mathcal{R}_{\mathbf{u}}$ and

$$\int_{\partial B_1} \Big| \frac{\mathbf{u}(x_1 + rx)}{r^\kappa} - \alpha \max(x\cdot \nu(x_1), 0)^\kappa \mathbf{e}(x_1) \Big| \, d\mathcal{H}^{n-1} \le C r^{\Lambda/2}, \tag{19}$$

for every $x_1 \in \partial\{|\mathbf{u}| > 0\} \cap \overline{B_{\delta_0}(x_0)}$ and for every $r \le \min(\delta_0, r_0)$. We now observe that $x_1 \mapsto \nu(x_1)$ and $x_1 \mapsto \mathbf{e}(x_1)$ are Hölder continuous with exponent β on $\partial\{|\mathbf{u}| > 0\} \cap \overline{B_{\delta_1}(x_0)}$ for some $\delta_1 \in (0, \delta_0)$:

$$\begin{aligned}
&\alpha \int_{\partial B_1} \big| \max(x\cdot\nu(x_1), 0)^\kappa \mathbf{e}(x_1) - \max(x\cdot\nu(x_2), 0)^\kappa \mathbf{e}(x_2) \big| \, d\mathcal{H}^{n-1} \\
&\le 2Cr^{\Lambda/2} + \int_{\partial B_1} \int_0^1 \Big| \frac{\nabla \mathbf{u}(x_1 + rx + t(x_2 - x_1))}{r^\kappa} \Big| |x_1 - x_2| \, dt d\mathcal{H}^{n-1} \\
&\le 2Cr^{\Lambda/2} + C_1 \frac{|x_1 - x_2|}{r^\kappa} \le (2C + C_1)|x_1 - x_2|^{\gamma\Lambda/2},
\end{aligned}$$

if we choose $\gamma := (\kappa + \frac{\Lambda}{2})^{-1}$ and $r := |x_1 - x_2|^\gamma \le \min(\delta_0, r_0)$, and the left hand side

$$\begin{aligned}
&\alpha \int_{\partial B_1} \big| \max(x\cdot\nu(x_1), 0)^\kappa \mathbf{e}(x_1) - \max(x\cdot\nu(x_2), 0)^\kappa \mathbf{e}(x_2) \big| \, d\mathcal{H}^{n-1} \\
&\qquad \ge c(n)(|\nu(x_1) - \nu(x_2)| + |\mathbf{e}(x_1) - \mathbf{e}(x_2)|)
\end{aligned} \tag{20}$$

as can be shown by an indirect argument. Suppose towards a contradiction that $c_j := |\nu_j^1 - \nu_j^2| + |\mathbf{e}_j^1 - \mathbf{e}_j^2| \to 0$, $(\nu_j^1 - \nu_j^2)/c_j \to \eta$, $(\mathbf{e}_j^1 - \mathbf{e}_j^2)/c_j \to \xi$, $\nu_j^1 \to \bar\nu$, $\mathbf{e}_j^2 \to \bar{\mathbf{e}}$

and

$$
\begin{aligned}
0 \leftarrow & \frac{1}{c_j}\int_{\partial B_1} \big|\max(x\cdot\nu_j^1,0)^\kappa \mathbf{e}_j^1 - \max(x\cdot\nu_j^2,0)^\kappa \mathbf{e}_j^2\big|\, d\mathcal{H}^{n-1}\\
\geq & \frac{1}{c_j}\int_{\partial B_1\cap\{x\cdot\nu_j^1>0\}\cap\{x\cdot\nu_j^2>0\}} \big|(x\cdot\nu_j^1)^\kappa(\mathbf{e}_j^1-\mathbf{e}_j^2) - \big((x\cdot\nu_j^1)^\kappa - (x\cdot\nu_j^2)^\kappa\big)\mathbf{e}_j^2\big|\, d\mathcal{H}^{n-1}\\
\to & \int_{\partial B_1\cap\{x\cdot\bar\nu>0\}} \big|(x\cdot\bar\nu)^\kappa\xi - \kappa(x\cdot\bar\nu)^{\kappa-1}(x\cdot\eta)\bar{\mathbf{e}}\big|\, d\mathcal{H}^{n-1}.
\end{aligned}
$$

Then $(x\cdot\bar\nu)\xi = \kappa(x\cdot\eta)\bar{\mathbf{e}}$ for all $x\in\partial B_1\cap\{x\cdot\bar\nu>0\}$. Putting $x=\bar\nu$ and noticing that $0 = (|\nu_j^1|^2 - |\nu_j^2|^2)/c_j \to 2\eta\cdot\bar\nu$, it follows that $\xi = 0$. So, $x\cdot\eta = 0$ for all x implying that $\eta = 0$. But $|\xi|+|\eta| = 1$. This contradiction proves (20).

Next, (19) as well as the regularity and nondegenracy of $\mathbf{u}$ imply that for $\epsilon>0$ there exists $\delta_2\in(0,\delta_1)$ such that for $x_1\in\partial\{|\mathbf{u}|>0\}\cap\overline{B_{\delta_1}(x_0)}$ and $y\in\overline{B_{\delta_2}(x_1)}$,

$$
\begin{aligned}
&\mathbf{u}(y) = 0 \quad \text{if} \quad (y-x_1)\cdot\nu(x_1) < -\epsilon|y-x_1|,\\
&|\mathbf{u}(y)| > 0 \quad \text{if} \quad (y-x_1)\cdot\nu(x_1) > \epsilon|y-x_1|.
\end{aligned}
\tag{21}
$$

Assuming that (21) does not hold, we obtain a sequence $\partial\{|\mathbf{u}|>0\}\cap\overline{B_{\delta_1}(x_0)}\ni x_m\to\bar x$ and a sequence $y_m - x_m\to 0$ as $m\to\infty$ such that

$$
\begin{aligned}
&\text{either } \mathbf{u}(y_m) = 0 \quad \text{and} \quad (y_m-x_m)\cdot\nu(x_m) < -\epsilon|y_m-x_m|,\\
&\text{or } |\mathbf{u}(y_m)| > 0 \quad \text{and} \quad (y_m-x_m)\cdot\nu(x_m) > \epsilon|y_m-x_m|.
\end{aligned}
\tag{22}
$$

On the other hand we know from (19) as well as from the regularity and nondegeneracy of the solution $\mathbf{u}$, that the sequence $\mathbf{u}_j(x) := \frac{\mathbf{u}(x_j+|y_j-x_j|x)}{|y_j-x_j|^\kappa}$ converges in $C^{1,\alpha}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ to $\alpha\max(x\cdot\nu(\bar x),0)^\kappa\mathbf{e}(\bar x)$ as $j\to\infty$ and that $\mathbf{u}_j = 0$ on each compact subset of $\{x\cdot\nu(\bar x)<0\}$ provided that $j\geq j(C)$. This, however, contradicts (22) for large j.

Last, we use (21) in order to show that $\partial\{|\mathbf{u}|>0\}$ is for some $\delta_3\in(0,\delta_2)$ in $\overline{B_{\delta_3}(x_0)}$ the graph of a differentiable function. Applying two rotations we may assume that $\nu(x_0) = \mathbf{e}_n$ and $\mathbf{e}(x_0) = \mathbf{e}_1$. Choosing now δ_2 with respect to $\epsilon = \frac{1}{2}$ and defining functions g^+, $g^-: B'_{\frac{\delta_2}{2}}(0)\to[-\infty,\infty]$,

$$
\begin{aligned}
g^+(x') &:= \sup\{x_n : x_0 + (x',x_n)\in\partial\{|\mathbf{u}|>0\}\}, \text{ and}\\
g^-(x') &:= \inf\{x_n : x_0 + (x',x_n)\in\partial\{|\mathbf{u}|>0\}\},
\end{aligned}
$$

we infer from (21) as well as from the continuity of $\nu(x)$ immediately that $\{x_n : x_0+(x',x_n)\in\partial\{|\mathbf{u}|>0\}\}$ is non-empty and that for sufficiently small δ_3 the functions g^+ and g^- are Lipschitz continuous and satisfy $g^+ = g^-$ on $\overline{B'_{\delta_3}(0)}$. Applying (21) once more with respect to arbitrary ϵ we see that g^+ is Fréchet-differentiable in $\overline{B'_{\delta_3}(0)}$, which finishes our proof in view of the already derived Hölder continuity of the normal vector $\nu(x)$. □

5. A system with Hölder coefficients

In this section we are going to show that our results extend to the setting

$$
\Delta\mathbf{u} = f(x,\mathbf{u}) := \lambda_+(x)|\mathbf{u}^+|^{q-1}\mathbf{u}^+ - \lambda_-(x)|\mathbf{u}^-|^{q-1}\mathbf{u}^-, \qquad \text{in } B_1(0),
\tag{23}
$$

where $\mathbf{u}^\pm = (u_1^\pm,\cdots,u_m^\pm)$ and $u_i^\pm := \max(\pm u_i, 0)$. Also, we assume that coefficients $\lambda_\pm$ satisfy

$$
0 < \lambda_0 \leq \lambda_\pm \in C^{0,\beta}(B_1).
$$

In fact, $\mathbf{u}$ is a minimizer of

$$J(\mathbf{u}) := \int_{B_1} |\nabla \mathbf{u}|^2 + 2F(x, \mathbf{u})\, dx,$$

where $F(x, \mathbf{u}) := \frac{1}{1+q}(\lambda_+(x)|\mathbf{u}^+|^{q+1} + \lambda_-(x)|\mathbf{u}^-|^{q+1})$.

First of all, we see that first part of Theorem 1.2 (case $\kappa \notin \mathbb{N}$) is valid. Even in the case $\kappa \in \mathbb{N}$, we may use its result to obtain the estimate

$$\sup_{B_r(z)} |\mathbf{u}| \le C_\varepsilon r^{\kappa - \varepsilon},$$

for every $\varepsilon > 0$. (Remark 2.4) It implies the monotonicity, Proposition 2.3, for the energy

$$W_s(\mathbf{u}, x_0, r) = \frac{1}{r^{n+2\kappa-2}} \int_{B_r(x_0)} \big(|\nabla \mathbf{u}|^2 + 2F_s(x, \mathbf{u})\big)\, dx - \frac{\kappa}{r^{n+2\kappa-1}} \int_{\partial B_r(x_0)} |\mathbf{u}|^2\, d\mathcal{H}^{n-1},$$

where $F_s(x, \mathbf{u}) := F(x_0 + s(x - x_0), \mathbf{u})$. We have the relation $W_s(r, x_0, \mathbf{u}_r) = W_1(rs, x_0, \mathbf{u})$ and the following (almost) monotonicity. (see [5] for similar setting but in the scalar case.)

Proposition 5.1. *Let $\mathbf{u}$ be a solution of (23) in $B_{r_0}(x_0)$ and $x_0 \in \Gamma^\kappa(\mathbf{u})$. There exist constants $C > 0$ and $\mu > 0$ such that $W_1(\mathbf{u}, x_0, r) + cr^\mu$ is increasing for $r > 0$.*

By the monotonicity we can repeat the second part of the proof of Theorem 1.2. Moreover, we have still non-degeneracy property as we have shown in Proposition 4.1. When coefficients $\lambda_\pm$ are constant, we can repeat the proof of Theorem 3.1 to show the Epiperimetric inequality for the energy function

$$M_{x_0}(\mathbf{v}) := \int_{B_1} \big(|\nabla \mathbf{v}|^2 + 2F(x_0, \mathbf{v})\big)\, dx - \kappa \int_{\partial B_1} |\mathbf{v}|^2 d\mathcal{H}^{n-1}.$$

We now proceed with the proof of the regularity of free boundary at regular points.

Theorem 5.2 (Energy decay)**.** *Let $x_0 \in B_1 \cap \partial\{|\mathbf{u}| > 0\}$, and suppose that the epiperimetric inequality holds with $\varepsilon \in (0, 1)$ for each*

$$\mathbf{c}_r(x) := |x|^\kappa \mathbf{u}_r\Big(\frac{x}{|x|}\Big) = \frac{|x|^\kappa}{r^\kappa} \mathbf{u}\Big(x_0 + \frac{r}{|x|} x\Big)$$

and for all $r \le r_0 < 1$. Finally let $\mathbf{u}_0$ denote an arbitrary blow-up limit of $\mathbf{u}$ at x_0 and $\Lambda = \min\{(n + 2\kappa - 2)\varepsilon/(1 - \varepsilon), \beta\}$. Then there exists a constant C such that

$$|W_1(\mathbf{u}, x_0, r) - W_1(\mathbf{u}, x_0, 0+)| \le C r^\Lambda |\log r|,$$

and

$$\int_{\partial B_1} |\mathbf{u}_r(x) - \mathbf{u}_0(x)| d\mathcal{H}^{n-1} \le C r^{\Lambda/2} |\log r|,$$

for all $r \in (0, r_0)$.

Proof. We can repeat the calculation in the proof of Theorem 4.2 to show that

$$e'(r) \ge -\frac{n + 2\kappa - 2}{r} \big(e(r) + W_1(\mathbf{u}, x_0, 0+) - W_r(\mathbf{c}_r, x_0, 1)\big).$$

Now we apply the epiperimetric inequality to $\mathbf{c}_r$ and find a function $\mathbf{v} \in W^{1,2}(B_1; \mathbb{R}^m)$ such that

$$M_{x_0}(\mathbf{v}) \le (1 - \varepsilon) M_{x_0}(\mathbf{c}_r) + \varepsilon M_{x_0}(\mathbf{h}).$$

Moreover, we may assume that

$$M_{x_0}(\mathbf{v}) \le M_{x_0}(\mathbf{u}_r), \tag{24}$$

otherwise we substitute $\mathbf{v}$ by $\mathbf{u}_r$. In order to replace $\mathbf{v}$ by $\mathbf{u}_r$ generally, we find the following estimate by the Hölder regularity assumption on $F(x,\mathbf{v})$ with respect to the variable x:

$$|M_{x_0}(\mathbf{v}) - W_r(\mathbf{v},x_0,1)| \le C_1 r^\beta \|\mathbf{v}\|^{1+q}_{\mathcal{L}^{1+q}(B_1)}, \tag{25}$$

for some constant C depending only on the coefficients of the problem. Freezing the coefficients and estimate (25) yields that

$$\begin{aligned} M_{x_0}(\mathbf{v}) &\ge W_r(\mathbf{v},x_0,1) - C_1 r^\beta \|\mathbf{v}\|^{1+q}_{\mathcal{L}^{1+q}(B_1)} \\ &\ge W_r(\mathbf{v},x_0,1) - \frac{1}{2}C_1 r^\beta (M_{x_0}(\mathbf{v}) + \kappa \|\mathbf{v}\|^2_{\mathcal{L}^2(\partial B_1)}) \\ &\ge W_r(\mathbf{v},x_0,1) - \frac{1}{2}C_1 r^\beta (M_{x_0}(\mathbf{u}_r) + \kappa \|\mathbf{u}_r\|^2_{\mathcal{L}^2(\partial B_1)}), \end{aligned}$$

where we have used (24) in the last line. Now by the minimality of $\mathbf{u}_r$ with the respect of its boundary conditions, we have that

$$\begin{aligned} e'(r) \ge &- \frac{n+2\kappa-2}{r}\big(e(r) + M_{x_0}(\mathbf{h}) - M_{x_0}(\mathbf{c}_r) + C_1 r^\beta \|\mathbf{c}_r\|^{1+q}_{\mathcal{L}^{1+q}(B_1)}\big) \\ \ge &- \frac{n+2\kappa-2}{r}\Big(e(r) + \frac{M_{x_0}(\mathbf{h}) - M_{x_0}(\mathbf{v})}{1-\varepsilon} + C_1 r^\beta \|\mathbf{c}_r\|^{1+q}_{\mathcal{L}^{1+q}(B_1)}\Big) \\ \ge &- \frac{n+2\kappa-2}{r}\Big(e(r) + \frac{1}{1-\varepsilon}\Big[M_{x_0}(\mathbf{h}) - W_r(\mathbf{v},x_0,1) + \frac{1}{2}C_1 r^\beta \big(M_{x_0}(\mathbf{u}_r) \\ &\quad + \kappa \|\mathbf{u}_r\|^2_{\mathcal{L}^2(\partial B_1)}\big)\Big] + C_1 r^\beta \|\mathbf{c}_r\|^{1+q}_{\mathcal{L}^{1+q}(B_1)}\Big) \\ \ge &- \frac{n+2\kappa-2}{r}\Big[e(r) + \frac{1}{1-\varepsilon}\big(M_{x_0}(\mathbf{h}) - W_r(\mathbf{u}_r,x_0,1)\big) + C_2 r^\beta\Big] \\ = &\frac{n+2\kappa-2}{r}\frac{\varepsilon}{1-\varepsilon}e(r) - C_3 r^{\beta-1} \ge \Lambda e(r) - C_3 r^{\Lambda-1}. \end{aligned}$$

Therefore

$$(r^{-\Lambda} e(r))' \ge -C_3 r^{-1},$$

and after integrating in (r,r_0), we get

$$e(r) \le C r^\Lambda |\log r|.$$

The second part of theorem can be obtained similarly to the proof of Theorem 4.2. □

Remark 5.3. *As apparent from the proof, we need the term* $\log r$ *in the estimate in Theorem 5.2 only when* $\beta = (n+2\kappa-2)\varepsilon/(1-\varepsilon)$*. Otherwise, the estimations are valid without this term.*

Other results in Section 4 remain true when we replace the equation with (23), especially Theorem 1.4 on the regularity of the free boundary.

Remark 5.4. *In this section we can treat case* $q = 0$ *(or* $\kappa = 2$*). Although, the proof of epiperimetric inequality in Section 3 needs the condition* $0 < q < 1$*, we already know the similar result in* [2]*. Then our proof in Theorem 5.2 covers the case* $q = 0$*.*

6. Appendix: Classification of global solutions in plane

We are going to classify the homogeneous solutions of degree κ of (3) in plane.

Proposition 6.1. *Let $n = 2$. If $\mathbf{u}$ is a homogeneous solutions of degree κ of (3) such that $\{x : |\mathbf{u}(x)| = |\nabla \mathbf{u}(x)| = 0\} \neq \{0\}$, then there exist a unit vector $\nu \in \mathbb{R}^2$ and vectors $\mathbf{e}_+, \mathbf{e}_- \in \mathbb{R}^m$ such that $|\mathbf{e}_\pm| = 0$ or 1, and*

$$\mathbf{u}(x) = \alpha \max(x \cdot \nu, 0)^\kappa \mathbf{e}_+ + \alpha \max(-x \cdot \nu, 0)^\kappa \mathbf{e}_-.$$

Proof. Suppose $\mathbf{u}(x) = r^\kappa \Phi(\theta)$, and rewrite the equation in polar coordinates. Then

$$\Phi'' + \kappa^2 \Phi = f(\Phi), \quad \Phi(0) = \Phi(2\pi).$$

Use polar coordinates for $\mathbb{R}^m$ and write $\Phi(\theta) = \rho(\theta)\hat{\Phi}(\theta)$, where $\hat{\Phi}$ is a unit vector in $\mathbb{R}^m$. According to the assumption there is θ_0 such that $\Phi(\theta_0) = \Phi'(\theta_0) = 0$. By a translation we may consider $(0, a)$ to be the maximal interval in which $|\Phi(\theta)| \neq 0$ and $\Phi(0) = \Phi'(0) = 0$. Then ρ and $\hat{\Phi}$ are smooth in $(0, a)$ and satisfy

$$\rho''\hat{\Phi} + 2\rho'\hat{\Phi}' + \rho\hat{\Phi}'' + \kappa^2\rho\hat{\Phi} = \rho^q\hat{\Phi}. \tag{26}$$

Now, using this fact that $\hat{\Phi} \cdot \hat{\Phi}' = 0$, and multiplying (26) in $\hat{\Phi}'$, we obtain

$$2\rho'|\hat{\Phi}'|^2 + \rho\hat{\Phi}'' \cdot \hat{\Phi}' = 0.$$

Thus

$$\frac{d}{d\theta}(\rho^4|\hat{\Phi}'|^2) = 0,$$

and $\rho^4|\hat{\Phi}'|^2$ is a constant function in interval $(0, a)$. On the other hand, $\rho(0) = |\Phi(0)| = 0$, and

$$\rho\hat{\Phi}' = \Phi' - (\hat{\Phi} \cdot \Phi')\hat{\Phi}$$

is bounded in $(0, a)$. Then $\lim_{\theta \to 0} \rho^4|\hat{\Phi}'|^2 = 0$, hence $\rho^4|\hat{\Phi}'|^2 \equiv 0$ for $\theta \in (0, a)$ and so $\hat{\Phi}$ is a constant vector in this interval since $\rho > 0$. (Note that $\hat{\Phi}$ is smooth as long as $\Phi \neq 0$.)

Therefore ρ must satisfy the following equation:

$$\rho'' + \kappa^2\rho = \rho^q \quad \text{in } (0, a).$$

Since $\rho(0) = \rho'(0) = 0$, according to Proposition 3.2 in [4], $a = \pi$ and $\rho(\theta) = \alpha \sin^\kappa(\theta)$. Then $\Phi(\pi) = 0$. Although ρ' is not necessarily continuous at $\theta = \pi$, Φ is smooth and

$$\Phi'(\pi) = \lim_{\theta \to \pi^-} \Phi'(\theta) = \lim_{\theta \to \pi^-} \rho'(\theta)\hat{\Phi} = 0.$$

Now we can repeat the above argument for another maximal interval $(b, c) \subseteq (\pi, 2\pi)$, such that $\Phi(b) = \Phi'(b) = 0$. We thus conclude that either $\rho = 0$ in $(\pi, 2\pi)$ or $(b, c) = (\pi, 2\pi)$ and $\rho(\theta) = \alpha \sin^\kappa(\theta - \pi)$. Therefore there are two unit vectors $\hat{\Phi}_+$ and $\hat{\Phi}_-$ such that

$$\mathbf{u}(x) = \alpha(x_1^+)^\kappa\hat{\Phi}_+ + \alpha(x_1^-)^\kappa\hat{\Phi}_-.$$

□

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