REMARKS ON HYPERBOLIC SCALING LIMITS

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1. INTRODUCTION

All microscopic models of hydrodynamics are constructed in such a way that a family of equilibrium (Gibbs) states is associated to the conservation laws of the underlying system; there is a one-to-one correspondence between the parameters of the Gibbs measures and the expected values of the conserved quantities. In theoretical physics the derivation of the macroscopic (Euler) equations is usually based on the **principle of local equilibrium**. This extremely strong form of the **ergodic hypothesis** allows us to evaluate the macroscopic fluxes as the expectation of the microscopic currents with respect to an equilibrium distribution with parameters depending on space and time. A mathematical manifestation of this approach requires an **identification of translation invariant stationary measures of the infinitely extended system as weak limits of superpositions of equilibrium Gibbs states with different parameters**. As a consequence, the skeleton (lattice approximation) of the macroscopic equations can be recognized in the fairly complex expression of the microscopic dynamics. The next step consists in the treatment of the remainders, and from this point techniques of PDE theory apply.

The principal microscopic models are coming from statistical mechanics. The strong ergodic hypothesis for such deterministic systems is certainly one of the hardest unsolved problems of mathematics, therefore apart from some exactly solvable models as hard rods or the harmonic chain [4,10], time dependent random driving effects are postulated to ensure the proper ergodic behavior of the microscopic system.

Hydrodynamic limit means that the space and time are simultaneously rescaled, see C. Morrey [22] for a first discussion. More precisely, if $0 < \varepsilon \to 0$ denotes the value of the microscopic unit of space at the

macroscopic level, then the microscopic time t is speeded up as $t = \tau/\varepsilon$ if the scaling rule is hyperbolic, that is of Euler type, while $t = \tau/\varepsilon^2$ in the diffusive case; here τ denotes the macroscopic time. Of course, the scaling rule is determined by the underlying microscopic model; the hyperbolic scaling limit of a diffusive system is trivial. Diffusive systems exhibit a parabolic structure, thus energy or entropy inequalities can be used to pass to the limiting parabolic equations see [11] and [19]. Due to the pioneering work [19] by Guo - Papanicolau - Varadhan, and [29] by S. R. S. Varadhan, the theory of diffusive scaling limit is more or less complete.

The case of hyperbolic models is much more difficult because the system has simply not enough time to develop local equilibrium. At a technical level we see that the viscosity, which has been created by postulating random effects, does vanish as the scaling parameter ε goes to 0. Consequently in contrast to diffusive models, there is no hope to get compactness in a standard way. Moreover, the macroscopic equations are also hyperbolic, thus shock waves may appear in a finite time. That is why a synthesis of probabilistic and PDE methods is needed here; a stochastic theory of compensated compactness seems to be the only effective tool in the case of systems of conservation laws. This approach is based on the evaluation of entropy production for Lax entropy pairs (additional conservation laws) of the macroscopic (Euler) equations; the first crucial step consists in the derivation of the celebrated Div-Curl Lemma. Since the strong ergodic hypothesis excludes the existence of nontrivial conservative quantities, the dynamics of a general Lax entropy exhibits an extremely turbulent, non-gradient behavior. We have to assume that the artificial viscosity of the model is strong enough to control these fluctuations. A logarithmic Sobolev inequality is our main tool at this step of the argument.

In this draft we are going to discuss some related open problems as the uniqueness of the scaling limit, the necessary strength of the artificial viscosity, and the derivation of the compressible Euler equations with viscosity.

2. An Asymmetric System of First Order

Let $V \in C^2(\mathbb{R})$ denote a not necessarily symmetric potential such that V'' is bounded and $V(y)/|y| \to +\infty$ as $y \to \pm\infty$. The infinite system

$$\dot{\eta}_k = \frac{1}{2}\left(V'(\eta_{k+1}) - V'(\eta_{k-1})\right), \quad \eta_k \in \mathbb{R}, k \in \mathbb{Z} \qquad (2.1)$$

is a direct lattice approximation to the conservation law $\partial_t y = \partial_x V'(y)$. Random perturbations of this model were studied in [15], see also [3]. It is easy to see that this system is uniquely solved in the space of configurations with a sub-exponential growth. Hyperbolic scaling means that we are interested in the asymptotic behavior of $y_\varepsilon(t,x) := \eta_k(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$ as $0 < \varepsilon \to 0$. Since the current of η_k is just $(-1/2)(V'(\eta_k) + V'(\eta_{k-1}))$, it is possible to imagine that y_ε converges is a weak sense to a solution to $\partial_t y = \partial_x V'(y)$, but the situation is much more complex.

Indeed, (2.1) is not a correct approximation procedure because it does not produce any effect of viscosity, therefore it is not easy to believe in its convergence; the principles of statistical physics provide a deeper insight into this issue. First of all, observe that there is another conservation law, namely that of $H = \sum V(\eta_k)$ with flux $(-1/2)\left(V'(\eta_{k-1})V'(\eta_k)\right)$ as

$$\partial_t V(\eta_k) = \frac{1}{2}\left(V'(\eta_k)V'(\eta_{k+1}) - V'(\eta_{k-1})V'(\eta_k)\right),$$

therefore the hydrodynamic limit should result in a couple of macroscopic equations. Let us remark that (2.1) can be interpreted as a Hamiltonian dynamics, see [3]. Indeed, introducing $p_k := \eta_{2k}$ and $r_k := \eta_{2k+1}$ for all $k \in \mathbb{Z}$, (2.1) turns into

$$\dot{p}_k = \frac{1}{2}\left(V'(r_k) - V'(r_{k-1})\right), \quad \dot{r}_k = \frac{1}{2}\left(V'(p_{k+1}) - V'(p_k)\right);$$

the associated Hamiltonian is just $H = (1/2)\sum\left(V(p_k) + V(r_k)\right)$.

Due to the existence of two conservation laws, for any chemical potential $\gamma \in \mathbb{R}$ and inverse temperature $\beta > 0$, the process (2.1) has a translation invariant stationary product measure $\lambda_{\beta,\gamma}$ with marginal Lebesque densities

$$f_{\beta,\gamma}(y) := \exp\left(\gamma y - \beta V(y) - F(\beta,\gamma)\right),$$

$$F(\beta,\gamma) := \log \int_{-\infty}^{\infty} \exp\left(\gamma y - \beta V(y)\right) \, dy.$$

By a direct computation we obtain that

$$\rho := \int \eta_k \, d\lambda_{\beta,\gamma} = F'_\gamma(\beta,\gamma), \quad \chi := \int V(\eta_k) \, d\lambda_{\beta,\gamma} = -F'_\beta(\beta,\gamma),$$

and $J := \int V'(\eta_k) \, d\lambda_{\beta,\gamma} = \gamma/\beta$. In view of the principle of local equilibrium, we expect $\partial_t \rho = \partial_x J$ and $\partial_t \chi = (1/2)\partial_x J^2$ as our couple of macroscopic equations.

To write J as a function of χ and ρ, we define the entropy function S of the problem as

$$S(\chi, \rho) := \sup_{\beta, \gamma} \{\gamma \rho - \beta \chi - F(\beta, \gamma) : \gamma \in \mathbb{R}, \beta > 0\},$$

whence by convex duality we get $\gamma = S'_\rho(\beta, \rho)$ and $\beta = -S'_\chi(\beta, \chi)$. Consequently the couple of macroscopic conservation laws reads as

$$\partial_t \rho(t, x) = \partial_x J(\chi, \rho), \quad \partial_t \chi(t, x) = (1/2)\partial_x J^2(\chi, \rho); \qquad (2.2)$$

where $J(\chi, \rho) = -S'_\rho(\chi, \rho)/S'_\chi(\chi, \rho)$. We see also that

$$\partial_t S(\chi, \rho) = S'_\rho(\chi, \rho)\partial_x J(\chi, \rho) + S'_\chi(\chi, \rho)J(\chi, \rho)\partial_x J(\chi, \rho) = 0$$

along classical solutions, which means that S is a convex Lax entropy with vanishing flux. This observation plays a role later on: in the presence of viscosity the space integral of S happens to be a decreasing function of time.

Deterministic artificial viscosity: We have several choices, perhaps the simplest one reads as

$$\dot{\eta}_k = \frac{1}{2}\left(V'(\eta_{k+1}) - V'(\eta_{k-1})\right) + \sigma\left(\eta_{k+1} + \eta_{k-1} - 2\eta_k\right), \qquad (2.3)$$

where $\sigma > 0$ is a large constant. This system does not admit locally absolutely continuous stationary states, the temperature is fixed at zero. If V is convex then the effect of viscosity is demonstrated by the following inequality:

$$\begin{aligned}
\partial_t H = \frac{1}{2}\sum_{k\in\mathbb{Z}} V'(\eta_k)\left(V'(\eta_{k+1} - V'(\eta_{k-1}))\right) \\
+ \sigma \sum_{k\in\mathbb{Z}} V'(\eta_k)\left(\eta_{k+1} + \eta_{k-1} - 2\eta_k\right) \\
= -\sigma \sum_{k\in\mathbb{Z}}\left(\eta_{k+1} - \eta_k\right)\left(V'(\eta_{k+1} - V'(\eta_k))\right) \leq 0,
\end{aligned}$$

which is also an effective a priori bound.

Since the conservation law of H is violated by the viscosity, the only macroscopic equation reads as $\partial_t y = (1/2)\partial_x V'(y)$, its derivation follows the classical argument of Olga Oleinik, see e.g. [20]. For example, with some intermediate values θ we have

$$\dot{\eta}_k + 2\sigma\eta_k = (\sigma + V''(\theta_{k+1})/2)\eta_{k+1} + (\sigma - V''(\theta_{k-1})/2)\eta_{k-1},$$

thus multiplying both sides by $e^{2\sigma t}$, we can estimate $\sup e^{2\sigma t}\eta_k$, i.e. $\sup \eta_k$, and also $\inf \eta_k$ by means of the Grönwall inequality, at lest if $0 < V'' < \sigma$.

The continuous dependence on initial values can be proven in a similar way, we get

$$\sum_{|k|<r} |\eta_k(t) - \bar{\eta}_k(t)| \leq \sum_{|k|<r+2\sigma t} |\eta_k(0) - \bar{\eta}_k(0)|$$

for two solutions. Oleinik's famous entropy inequality follows from the expansion

$$\dot{\xi}_k + 2\sigma\xi_k = (\sigma + V''(\eta_k)/2)\xi_{k+1} - (1/4)V'''(\theta_{k+1})\xi_{k+1}^2$$
$$+ (\sigma - V''(\eta_k)/2)\xi_{k-1} - (1/4)V'''(\theta_{k-1})\xi_{k-1}^2$$

for $\xi_k := \eta_{k+1} - \eta_{k-1}$, where $V''' > 0$ is a relevant condition.

Suppose that the initial configuration is bounded, then the summary of these computations yields L^1 convergence of the scaled process to the unique entropy solution to the macroscopic equation. In contrast to the stochastic model of [15], it is not necessary to assume that the microscopic viscosity $\sigma = \sigma(\varepsilon)$ goes to infinity; it must be a large constant depending on the initial condition. Concerning the unbounded case we claim that it is sufficient to assume that $\sigma \to +\infty$ in a moderate way, $\varepsilon\sigma^2(\varepsilon) \to +\infty$ is not required. A stochastic theory is to be discussed later.

The case of energy: There are not too many choices. The viscid perturbation

$$\dot{\eta}_k = \frac{1}{2}\left(V'(\eta_{k+1}) - V'(\eta_{k-1})\right) \tag{2.4}$$
$$+ (\sigma/V'(\eta_k))\left(V'(\eta_{k+1}) + V'(\eta_{k-1}) - 2V'(\eta_k)\right)$$

implies $\partial_t H = 0$ and

$$\partial_t \sum_{k\in\mathbb{Z}} \eta_k = -\sigma \sum_{k\in\mathbb{Z}} \frac{(V'(\eta_{k+1}) - V'(\eta_k))^2}{V'^2(\eta_k)}$$

as formal identities. The singular case of $V'(\eta_k)$ can be excluded by assuming $V'(y) < 0$ if $y > 0$ and $V(y) \to +\infty$ as $y \to 0$; lattice models of gas dynamics are driven by such interactions. It is possible to show that $\eta_k(t) > 0$ remains is force if $\eta_k(0) > 0$ for all $k \in \mathbb{Z}$ and $H < +\infty$ at time zero.

This version seems to be simpler then the Hamiltonian model of gas dynamics, but there are serious difficulties also here. First of all, V'' is the order of the Lipschitz factor of the right hand side of (2.4), which might be much bigger than V, thus it is not controlled by the energy bound. As a consequence, we can not prove the convergence of partial dynamics when the size of the system goes to infinity. Of course, the derivation of the single macroscopic equation $\partial_t\chi = (1/2)\partial_x W(\chi)$,

where $W(\chi) := V'^2(y)$ if $V(y) = \chi$, does not presuppose the existence of the infinitely extended dynamics, just it seems to be even harder.

3. The Anharmonic Chain

It is simple microscopic model of one-dimensional elasticity. The Hamiltonian of coupled oscillators of unit mass on $\mathbb{Z}$ reads as

$$H(\omega) := \sum_{k \in \mathbb{Z}} H_k(\omega), \quad H_k(\omega) := p_k^2/2 + V(q_{k+1} - q_k),$$

where $\omega = \{(p_k, q_k) : k \in \mathbb{Z}\}$ denotes a configuration of the infinite system, $p_k, q_k \in \mathbb{R}$ are the momentum (velocity) and the coordinate of the oscillator at site $k \in \mathbb{Z}$; the interaction potential V is the same as it was in the previous section. In terms of the deformation (strain) variables $r_k := q_{k+1} - q_k$, the equations of motion read as

$$\dot{p}_k = V'(r_k) - V'(r_{k-1}) \quad \text{and} \quad \dot{r}_k = p_{k+1} - p_k \quad \text{for } k \in \mathbb{Z}; \quad (3.1)$$

in this formulation V needs not be symmetric. The solutions can be well approximated by the solutions to finite subsystems when the size of the finite system goes to infinity, see e.g. [16] with further references. The basic questions on (3.1) are more or less the same as those on (2.1), the right way of its regularization is suggested by the small viscosity approach. Deterministic versions might also be interesting, but the theory of hydrodynamic limits goes beyond numerical analysis as discussed below.

Stationary states: (3.1) reads as a lattice system of conservation laws for the total momentum $P := \sum p_k$, and for the total deformation $R := \sum r_k$, respectively: $\partial_t P = \partial_t R = 0$ are formal identities. Since $\partial_t H_k(\omega) = p_{k+1} V'(r_k) - p_k V'(r_{k-1})$ is a difference of currents, total energy H is also preserved by the dynamics, therefore we expect to have three hydrodynamic equations: one for momentum, one for the deformation, and a third one for energy. In view of the principle of local equilibrium, the macroscopic fluxes of these conservative quantities are to be calculated by means of the stationary states of the dynamics. These are characterized by $\int \mathcal{L}_0 \varphi \, d\lambda = 0$ for smooth local functions φ of a finite number of variables, where

$$\mathcal{L}_0 \varphi := \sum_{k \in \mathbb{Z}} \left((V'(r_k) - V'(r_{k-1})) \frac{\partial \varphi}{\partial p_k} + (p_{k+1} - p_k) \frac{\partial \varphi}{\partial r_k} \right)$$

denotes the associated Liouville operator. Also by means of the finite volume approximation it is easy to check that, associated with the classical conservation laws, we have a three-parameter family $\lambda_{\beta,\pi,\gamma}$ of translation invariant stationary product measures. As before, $\beta > 0$

is the inverse temperature, $\gamma \in \mathbb{R}$ is a chemical potential and $\pi \in \mathbb{R}$ denotes the mean velocity. Under $\lambda_{\beta,\pi,\gamma}$ the marginal Lebesgue density of any couple $(p_k, r_k) \sim (y, z)$ reads as $\exp(\gamma z - \beta I(y, z|\pi) - F(\beta, \gamma))$, where $I(y, y|\pi) := (y - \pi)^2/2 + V(z)$; the normalization

$$F(\beta, \gamma) := \log \int_{\mathbb{R}^2} \exp\left(\gamma z - \beta I(y, z|\pi)\right) dy \, dz$$

is sometimes referred to as the free energy. It is easy to see that $\mathcal{L}_0$ is antisymmetric with respect to any $\lambda_{\beta,\pi,\gamma}$.

The compressible Euler equations: The expected values (χ, π, ρ) of the conservative quantities H_k, p_k and r_k with respect to $\lambda_{\beta,\pi,\gamma}$ read as $\chi := \mathfrak{I} + \pi^2/2$, where $\mathfrak{I} := \int I_k \, d\lambda_{\beta,\pi,\gamma} = -F_\beta'(\beta, \gamma)$ is the equilibrium mean of the internal energy $I_k := I(p_k, r_k|\pi)$ at site k, $\pi := \int p_k \, d\lambda_{\beta,\pi,\gamma}$, while $\rho := \int r_k \, d\lambda_{\beta,\pi,\gamma} = F_\gamma'(\beta, \gamma)$ is the mean deformation. Integrating by parts we obtain $\int V'(r_k) \, d\lambda_{\beta,\pi,\gamma} = \gamma/\beta$ for the equilibrium expectation of V'. The parameters β and γ can be expressed in terms of the thermodynamical entropy

$$S(\mathfrak{I}, \rho) := \sup\left\{\gamma\rho - \beta\mathfrak{I} - F(\beta, \gamma) : \beta > 0, \gamma \in \mathbb{R}\right\}$$

as follows. Since S is the convex conjugate of F, we have $\gamma = S_\rho'(\mathfrak{I}, \rho)$ and $\beta = -S_\mathfrak{I}'(\mathfrak{I}, \rho)$ if $v = F_\gamma'(\beta, \gamma)$ and $\mathfrak{I} = -F_\beta'(\beta, \gamma)$.

We are interested in the asymptotic behavior of the empirical processes $\pi_\varepsilon(t, x) := p_k(t/\varepsilon)$, $\rho_\varepsilon(t, x) := r_k(t/\varepsilon)$ and $\chi_\varepsilon(t, x) := H_k(\omega(t/\varepsilon))$ if $|\varepsilon k - x| < \varepsilon/2$, as $0 < \varepsilon \to 0$. Of course it is assumed that at time zero these processes converge, at least in a weak sense to the corresponding initial values of the hydrodynamic equations. In view of the physical principle of local equilibrium, the macroscopic currents of the conservative quantities should be calculated by means of a product measure of type $\lambda_{\beta,\pi,\gamma}$ with parameters depending on time and space. In this framework $J := \gamma/\beta = \int V'(r_k) \, d\lambda_{\beta,\pi,\gamma}$ is the mean current of momentum, and $\pi J = \int p_k V'(r_{k-1}) \, d\lambda_{\beta,\pi,\gamma}$ is the mean current of energy, consequently a formal calculation results in the triplet of compressible Euler equations:

$$\partial_t \pi = \partial_x J(\mathfrak{I}, \rho), \quad \partial_t \rho = \partial_x \pi \quad \text{and} \quad \partial_t \chi = \partial_x(\pi J(\mathfrak{I}, \rho)), \qquad (3.2)$$

where $J(\mathfrak{I}, \rho) := \gamma/\beta = -S_\rho'(\mathfrak{I}, \rho)/S_\mathfrak{I}'(\mathfrak{I}, \rho)$ and $\mathfrak{I} = \chi - \pi^2/2$, see e.g. [7] or [14]. Therefore $\partial_t \mathfrak{I} = J(\mathfrak{I}, \rho)\partial_x \pi$ and $\partial_t S(\mathfrak{I}, \rho) = 0$ along classical solutions, but we have to keep in mind that this system develops shock waves in a finite time.

4. Stochastic Perturbations

The first step of the derivation of the macroscopic conservation laws consists in the evaluation of the microscopic flux as its canonical expectation with respect to an equilibrium Gibbs state. This crucial step is based on the **strong ergodic hypothesis**, which means an identification of all translation invariant stationary measures as equilibrium Gibbs states. That is why the dynamics of the anharmonic chain should be regularized by a well chosen noise. There are several plausible tricks, we are going to consider Markov processes generated by an operator $\mathcal{L} = \mathcal{L}_0 + \sigma \mathcal{G}$, where $\mathcal{L}_0$ is the Liouville operator, while the Markov generator $\mathcal{G}$ is symmetric in equilibrium. Here $\sigma > 0$ may depend on the scaling parameter $\varepsilon > 0$, and $\varepsilon\sigma(\varepsilon)$ is interpreted as the **coefficient of macroscopic viscosity**. Following the **vanishing viscosity approach** of PDE theory, we are assuming that $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, then the effect of the symmetric component $\sigma\mathcal{G}$ diminishes in the limit. In a regime of shocks an additional technical condition: $\varepsilon\sigma^2(\varepsilon) \to +\infty$ is also needed.

Block averages: The first step of a proof is always the substitution of the currents by their macroscopic estimators. This is due to the strong ergodicity of the dynamics, and it is carried out in terms of block averages. For any sequence $\{\xi_k\}$ indexed by $\mathbb{Z}$ we write $\bar{\xi}_{l,k} := (1/l)(\xi_{k-l+1} + \xi_{k-l+2} + ... + \xi_k)$. For example, $\bar{V}'_{l,k}$ is the sequence of arithmetic means for $\{V'(r_k)\}$; the symbols $\bar{p}_{l,k}$, $\bar{p}_{l,k}$, $\bar{r}_{l,k}$ $\bar{H}_{l,k}$ are defined in a similar way, and so on. In case of the Euler equations we substitute $\bar{V}'_{l,k}$ by $J(\bar{\mathcal{J}}_{l,k}, \bar{r}_{l,k})$, where $\bar{\mathcal{J}}_{l,k} = \bar{H}_{l,k} - (1/2)(\bar{p}_{l,k})^2$, while the block averages of $p_k V'(r_{k-1})$ are estimated by $\bar{p}_{l,k} J(\bar{\mathcal{J}}_{l,k}, \bar{r}_{l,k})$. Depending on the problem, the empirical processes may also be defined in terms of averages of block size $l = l(\varepsilon)$ as $\bar{\pi}_\varepsilon(t, x) := \bar{p}_{l,k}(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$. Because of technical reasons, in a regime of shocks the "flat" arithmetic means $\bar{\xi}_{l,k}$ has to be replaced by the more smooth "triangular" averages $\hat{\xi}_{l,k}$, see a bit later.

Random exchange of velocities: This is a weak but still effective conservative noise: at neighboring sites we simply do an exchange of the two velocities in an independent way. More precisely, the generator $\mathcal{G} = \mathcal{G}_{ep}$ of this exchange mechanism is acting on local functions as

$$\mathcal{G}_{ep}\varphi(\omega) = \sum_{k \in \mathbb{Z}} \left(\varphi(\omega^{k,k+1}) - \varphi(\omega) \right), \qquad (4.1)$$

where $\omega^{k,k+1}$ denotes the configuration obtained from $\omega = \{(p_j, r_j)\}$ by exchanging p_k and p_{k+1}, the rest of ω remains unchanged, see [13]. It

is plain that $P = \sum p_k$, $R = \sum r_k$ and total energy H are formally preserved by $\mathcal{G}_{ep}$, and the product measures $\lambda_{\beta,\pi,\gamma}$ are all stationary states of the Markov process generated by $\mathcal{L} := \mathcal{L}_0 + \sigma\, \mathcal{G}_{ep}$ if $\sigma > 0$.

The relative entropy $\mathsf{S}[\mu|\lambda]$ of two probability measures is defined as $\mathsf{S} := \int \log f\, d\mu$ if $\mu \ll \lambda$, $f = d\mu/d\lambda$ and the integral exists; $\mathsf{S}[\mu|\lambda] = +\infty$ otherwise. Let μ_n denote the joint distribution of the variables $\{(p_k, r_k) : |k| \leq n\}$ with respect to μ, as a reference measure we choose $\lambda := \lambda_{1,0,0}$, and $f_n := d\mu_n/d\lambda$. Following [13] we see that if μ is a translation invariant stationary measure, and $\mathsf{S}[\mu_n|\lambda] = O(n)$ then μ is a superposition of our product measures $\{\lambda_{\beta,\pi,\gamma}\}$, see also [3].

A theory on the preservation of local equilibrium has been initiated by H.-T. Yau [30] as follows. At a level $\varepsilon = 1/n$, $n \in \mathbb{N}$ of scaling let $\mu_{t,n}$ denotes the true distribution of the variables $\{(p_k(tn), r_k(tn)) : |k| \leq n\}$. Since our noise is not strong enough to control boundary effects, we have to assume that our configurations are periodic with period n. The basic idea of his strategy is to fit a product measure $\lambda_{t,n} = \lambda_{\beta,\pi,\gamma}$ with space and time dependent parameters to $\mu_{t,n}$ in such a way that $\mathsf{S}[\mu_{t,n}|\lambda_{t,n}] = o(n)$ as long as possible. This is a condition at $t = 0$, and assuming that the initial condition of (3.2) determines a classical solutions on the interval $[0, T)$, the claim can be proven for $t < T$ by defining the parameters of $\lambda_{t,n}$ as they are predicted by the compressible Euler equations. This construction implies immediately the convergence of the empirical processes to the prescribed smooth solution of the Euler equations on $[0, T)$. In the temporal derivative of S the substitution of the microscopic currents by their canonical equilibrium expectations is done at the level of large block averages, but there are several other steps where the continuous differentiability of the macroscopic solution is exhausted. Since $\partial_t S$ is one of the leading terms of $\partial_t \mathsf{S}[\mu_{t,n}|\lambda_{t,n}]$, the identity $\partial_t S(\mathfrak{I}, \rho) = 0$ is very important, too. Regularization by exchange works in much the same way also in case of (2.1), we obtain by scaling the corresponding couple (2.2) of hyperbolic equations.

Let us remark that in a periodic setting the identification of the translation invariant stationary states is easier than in the general case. Indeed, we only have to describe those measures, which are obtained as weak limits of space - time averages of periodic processes.

Physical viscosity with thermal noise: The Ginzburg-Landau type perturbation of velocities:

$$
\begin{aligned}
dp_k &= \left(V'(r_k) - V'(r_{k-1})\right) dt + \sigma\left(p_{k+1} + p_{k-1} - 2p_k\right) dt \\
&\quad + \sqrt{2\sigma}\left(dw_k - dw_{k-1}\right), \quad dr_k = \left(p_{k+1} - p_k\right) dt, \quad k \in \mathbb{Z},
\end{aligned}
\tag{4.2}
$$

where $\sigma > 0$ is a given constant and $\{w_k : k \in \mathbb{Z}\}$ is a family of independent Wiener processes, maintains a thermal equilibrium at unit temperature and violates the law of energy conservation at the same time. Therefore the set of our stationary product measures reduces to $\{\lambda_{\pi,\gamma}\} := \{\lambda_{1,\pi,\gamma}\}$; the strong ergodic hypothesis holds true in this setting, too. Let $F(\gamma) := F(1,\gamma)$, then $\int V'(r_k)\,d\lambda_{\pi,\gamma} = \gamma = S'(\rho)$ if $\int r_k\,d\lambda_{\pi,\gamma} = \rho = F'(\gamma)$, where $S(\rho) := \sup\{\gamma\rho - F(\gamma) : \gamma \in \mathbb{R}\}$. Consequently (3.2) turns into the nonlinear sound equation of elastodynamics:

$$\partial_t \pi = \partial_x S'(\rho) \quad \text{and} \quad \partial_t \rho = \partial_x \pi, \quad \text{that is} \quad \partial_t^2 \rho = \partial_x^2 S'(\rho). \quad (4.3)$$

In a smooth regime the derivation of this p-system follows the relative entropy argument of Yau [30], cf. the paragraph above and [14]; in this case it is not necessary to assume that the configurations are periodic, other conditions are not changed. The thermodynamic entropy now reads as $\tilde{S} := \pi^2/2 + S(\rho)$. Of course $\tilde{S}$ is a convex Lax entropy as $\partial_t \tilde{S} = \partial_x(\pi S'(\rho))$, thus we have $\partial_t \int \tilde{S}\,dx = 0$ along smooth solutions.

At the end of the paper we are going to discuss two other models, which are also driven by Wiener processes, and the second one preserves the classical conservation laws including that of total energy, too.

HDL in a regime of shocks: The hyperbolic scaling limit of attractive models can be determined even in the presence of shocks, see F. Rezakhanlou [24] for a general argument. In fact, his effective coupling techniques reduce the problem to the Kruzkov entropy condition. If V is convex then

$$
\begin{aligned}
d\eta_k = &\frac{1}{2}\left(V'(\eta_{k+1}) - V'(\eta_{k-1})\right)dt \\
&+ \sigma\left(V'(\eta_{k+1}) + V'(\eta_{k-1}) - 2V'(\eta_k)\right)dt + \sqrt{2\sigma}\left(dw_k - dw_{k-1}\right)
\end{aligned}
$$

defines an attractive model, see [15] for its hyperbolic scaling limit.

Two-component systems as the anharmonic chain and its random perturbations are certainly not attractive, compensated compactness seems to be the only tool we can use. The microscopic dynamics can not admit non-classical conservation laws because it should be ergodic in the strong sense, therefore a nontrivial Lax entropy is not conserved by the microscopic dynamics. In general, the flux of a Lax entropy exhibits a non-gradient behavior, and the standard spectral gap estimates S. R. S. Varadhan [29] are not sufficient to bound the remainders, a logarithmic Sobolev inequality is needed. This effective LSI is due to the strong

artificial viscosity of our next model:

$$dp_k = (V'(r_k) - V'(r_{k-1})) \, dt + \sigma(\varepsilon) \, (p_{k+1} + p_{k-1} - 2p_k) \, dt$$
$$+ \sqrt{2\sigma(\varepsilon)} \, (dw_k - dw_{k-1})$$

and

$$dr_k = (p_{k+1} - p_k) \, dt + \sigma(\varepsilon) \, (V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k)) \, dt$$
$$+ \sqrt{2\sigma(\varepsilon)} \, (d\tilde{w}_{k+1} - d\tilde{w}_k) \, ,$$

where $\{w_k : k \in \mathbb{Z}\}$ and $\{\tilde{w}_k : k \in \mathbb{Z}\}$ are independent families of independent Wiener processes. Of course, the macroscopic viscosity $\varepsilon\sigma(\varepsilon)$ vanishes as $\varepsilon \to 0$, but we also need $\varepsilon\sigma^2(\varepsilon) \to +\infty$ to suppress extreme fluctuations of Lax entropies.

Just as in the case of (4.2), the same $\{\lambda_{\pi,\gamma} : \pi, \gamma \in \mathbb{R}\}$ is the family of stationary product measures, and the strong ergodic hypothesis also holds true. Therefore again (4.3) is expected to govern the macroscopic behavior of the system under hyperbolic scaling. Since we are not able to prove the uniqueness of the limit in a regime of shocks, we only assume that $\mathsf{S}[\mu_{0,\varepsilon,n}|\lambda_{0,0}] = O(n)$, where $\mu_{0,\varepsilon,n}$ denotes the initial distribution in the box $[-n, n]$. This bound remains in force also for the evolved measure, and the dominant part of $-\partial_t \mathsf{S}$ is a Dirichlet form $\mathsf{D} \geq 0$.

The evaluation of the Lax entropy production requires a modification of the empirical processes. Instead of the standard arithmetic means, the more smooth averages

$$\hat{\xi}_{l,k} := \frac{1}{l^2} \sum_{j=-l}^{l} (l - |j|) \, \xi_{k+j}$$

are used in the definition of $\hat{\pi}_\varepsilon$ and $\hat{\rho}_\varepsilon$. The computation of the microscopic currents is based on the following sharp a priori bound:

$$\sum_{|k|<n} \int_0^t \int \left(\bar{V}'_{l,k} - S'(\bar{r}_{l,k}) \right)^2 \, d\mu_{s,\varepsilon} \, ds \leq C \left(\frac{nt}{l} + \frac{l^2 \sqrt{n^2 + \sigma(\varepsilon)t}}{\sigma(\varepsilon)} \right) \, .$$

This is a consequence of our bound on D via the associated logarithmic Sobolev inequality, and it is really useful if $l = l(\varepsilon) = o(\sigma(\varepsilon))$, while $\sigma(\varepsilon) = o(\varepsilon l^3(\varepsilon))$.

Now we are in a position to extend the results of J. W. Shearer [27] and Serre - Shearer [26] on compensated compactness to our stochastic model, see [18]. Since the genuine nonlinearity of (4.3) is a keyword of the proof, we have to assume that even V' is convex, or V is symmetric and zero is the only root of V'''. Let us remark that the in case of the creation - annihilation model of our paper [1], the action of LSI we have

there should be supplemented by an additional tool of PDE theory: the method of relaxation schemes.

Local behavior of total variation: The most striking unsolved problem of this theory of hyperbolic scaling limits is certainly the uniqueness of the hydrodynamic limit in the case of two conservation laws as (4.3), say. The more or less general results of A. Bressan [5] on the uniqueness of the Cauchy problem for one - dimensional systems of conservation laws presuppose that a sophisticated Oleinik type entropy condition holds true. Unfortunately, our a priori bounds on the microscopic dynamics usually read as expectations of temporal integrals of spatial sums. Remember that hyperbolic scaling of the exceptional attractive models results in a single conservation law, in which case the question of uniqueness is much easier.

In some cases local bounds on the total variation imply the uniqueness of the Cauchy problem, let us discuss a bit this issue. The total variation of an empirical process $\hat{y}_\varepsilon(t,x) := \hat{\eta}_{l,k}(t/\varepsilon)$, $|\varepsilon k - x| < \varepsilon/2$ on $(-a, a)$ reads as

$$Tv(\hat{y}_\varepsilon, t, a) := \sum_{\varepsilon|k|<a} |\hat{\eta}_{l,k+1} - \hat{\eta}_{l,k}| = \frac{1}{l} \sum_{\varepsilon|k|<a} |\bar{\eta}_{l,k+l} - \bar{\eta}_{l,k}|.$$

Observe that $\bar{\eta}_{l,k+l} - \bar{\eta}_{l,k} = l^{-1/2}(N_k + \delta_k\sqrt{l})$, where N_k is a centered and normalized sum of $2l$ variables, while δ_k is the difference of the mean values. Suppose now that the underlying distribution of $\{\eta_k\}$ is a product (local equilibrium) measure with regularly varying parameters, that is $\delta_k \approx \varepsilon l$ when $\varepsilon \to 0$, and the variances are strictly positive and bounded. Since each N_k is asymptotically normal, we see three possibilities. $Tv(\eta_\varepsilon, t, a)$ explodes if $\varepsilon l^{3/2} \to 0$ as $\varepsilon \to 0$ because averages over small blocks are rapidly fluctuating. If $\varepsilon l^{3/2} \to +\infty$ then our blocks are too large, thus $Tv(\eta_\varepsilon, t, a) \to 0$ is expected in this case. Consequently $l = l(\varepsilon) \approx \varepsilon^{-2/3}$ is the only choice when the total variation behaves in a standard way.

Of course, it is rather exceptional that the evolved process is so close to local equilibrium that these considerations really apply. Anyway, to get a bound on the total variation, very large blocks of size $l = l(\varepsilon) \gg \varepsilon^{-2/3}$ should be considered. Since $l(\varepsilon) \ll \sigma(\varepsilon)$ when compensated compactness is used, in a regime of shocks the coefficient of artificial viscosity must be as big as $\sigma(\varepsilon) \gg \varepsilon^{-2/3}$; otherwise there is no hope to bound total variation. Let us remark that in PDE theory the Riemann invariants are examined in this context, and at the microscopic level this issue is fairly involved.

5. MACROSCOPIC VISCOSITY

Compensated compactness does not work in the case of three conservation laws, therefore it might be interesting to study microscopic systems with non - vanishing viscosity, see Chen - Dafermos [6] for discussions.

The p-system of thermoelasticity: Due to its viscid term, the system $\partial_t \rho = \partial_x \pi$ and $\partial_t \pi = \partial_x S'(\rho) + \sigma \partial_t^2 \pi$, $\sigma > 0$ is not scaling invariant; one of its microscopic version reads as

$$dp_k = (V'(r_k) - V'(r_{k-1}))\, dt + (\sigma/\varepsilon)\,(p_{k+1} + p_{k-1} - 2p_k)\, dt$$
$$+ \sqrt{2\sigma/\varepsilon}\,(dw_k - dw_{k-1})\,, \quad \dot{r}_k = p_{k+1} - p_k\,,$$

where $\{w_k\}$ is the usual Wiener family, cf. (4.2). The limiting behavior of this model can be described by a standard application of the relative entropy argument. As before, we have to assume that the macroscopic solution is smooth, but the configurations of the system need not be periodic, see [14].

Perturbation of deformations: The physical motivation of the next version is less convincing, but it might be more interesting from the point of view of mathematics. Let us consider

$$dr_k = (p_{k+1} - p_k)\, dt + \sigma(\varepsilon)\,(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k))\, dt$$
$$+ \sqrt{2\sigma/\varepsilon}\,(d\tilde{w}_{k+1} - d\tilde{w}_k)\,, \quad \dot{p}_k = V'(r_k) - V'(r_{k-1})\,,$$

where σ is a positive constant, and $\{w_k : k \in \mathbb{Z}\}$ is a family of standard Wiener processes. Here we are luckier than in the previous case because the artificial viscosity of the present model controls the differences of the nonlinear V' variables. In fact, an entropy argument allows us to extend the celebrated two - blocks estimate of Guo - Papanicolau - Varadhan [19] to get strong compactness in L^2. We even have an LSI, but it is not necessary for the derivation of $\partial_t \pi = \partial_t S'(\rho)$ and $\partial_t \rho = \partial_x \pi + \sigma\,\partial_x^2 S'(\rho)$ as the couple of macroscopic equations, see [12].

In view of the remarks above, it is reasonable to replace the intensity σ/ε of microscopic viscosity by $\sigma(\varepsilon)$ such that $\varepsilon\sigma(\varepsilon) \to 0$ but $\varepsilon\sigma^2(\varepsilon) \to +\infty$. Indeed, our fundamental a priori bound on $\bar{V}' - S'(\bar{r})$ follows as before via LSI, thus we are in a position to launch compensated compactness. Unfortunately some less crucial steps are not clear, nevertheless we conjecture that (4.3) follows from this model even is the presence of shocks.

The viscid Euler equations: The random exchange mechanism can also be speeded up to get viscosity at the macroscopic level. This means that we investigate the process with generator $\mathcal{L} := \mathcal{L}_0 + (\sigma/\varepsilon)\,\mathcal{G}_{ep}$, and

the method of Yau results in the triplet $\partial_t \rho = \partial_x \pi$,

$$\partial_t \pi = \partial_x J(\mathfrak{I}, \rho) + \sigma\, \partial_x^2 \pi\,,\ \ \partial_t \chi = \partial_x (\pi J(\mathfrak{I}, \rho)) + (\sigma/2)\, \partial_x^2 (\beta^{-1} + \pi^2)\,, \ \ (5.1)$$

where $(1/2)(\beta^{-1} + \pi^2)$ is the equilibrium mean of the kinetic energy, and $\beta = -S_\mathfrak{I}'(\mathfrak{I}, \rho)$, cf. (3.2).

Energy preserving stochastic equations: Basile - Bernardin - Olla [2] notice that the operators

$$\mathcal{Y}_k := (p_k - p_{k+1})\, \partial/\partial p_{k-1} + (p_{k+1} - p_{k-1})\, \partial/\partial p_k + (p_{k-1} - p_k)\, \partial/\partial p_{k+1}$$

preserve both the total momentum and the kinetic energy of three consecutive sites, namely

$$\mathcal{Y}_k p_{k-1} + \mathcal{Y}_k p_k + \mathcal{Y}_k p_{k+1} = 0 = \mathcal{Y}_k p_{k-1}^2 + \mathcal{Y}_k p_k^2 + \mathcal{Y}_k p_{k+1}^2\,.$$

Motivated by this simple fact, let us consider now the process generated by $\mathcal{L} := \mathcal{L}_0 + \sigma \mathcal{G}_{ch}$, where $\sigma = \sigma(\varepsilon) > 0$ and $\mathcal{G}_{ch}\varphi := (1/6) \sum_{k \in \mathbb{Z}} \mathcal{Y}_k^2\, \varphi$.

It is easy to check that $\mathcal{G}_{ch}$ is formally symmetric in each $L^2(\lambda_{\pi,\gamma})$, thus $\mathcal{L}$ preserves the classical conservation laws P, R, and H. Moreover, each product measure $\lambda_{\beta,\pi,\gamma}$ is stationary under the stochastic dynamics generated by $\mathcal{L}$, but the converse statement is not so clear for a first sight. The stochastic differential equations of motion read as $\dot{r}_k = p_{k+1} - p_k$ and

$$\begin{aligned}
dp_k &= \big(V'(r_k) - V'(r_{k-1})\big)\, dt \\
&\quad + (\sigma/6)(p_{k+2} + 2p_{k+1} + 2p_{k-1} + p_{k-2} - 6p_k)\, dt \\
&\quad + \sqrt{\sigma/3}\, \big((p_{k+1} - p_{k-1})\, dw_k + (p_{k+2} - p_{k+1})\, dw_{k+1}\big) \\
&\quad + \sqrt{\sigma/3}\, (p_{k-2} - p_{k-1})\, dw_{k-1}\,.
\end{aligned}$$

Observe now that by a direct calculation we get

$$6\mathcal{G}_{ch} p_k = 2(p_{k+1} + p_{k-1} - 2p_k) + (p_{k+2} + p_{k-2} - 2p_k)\,,$$

while

$$\begin{aligned}
3\mathcal{G}_{ch} p_k^2 &= (p_{k+1} - p_{k-1})^2 + p_k(p_{k+1} + p_{k-1} - 2p_k) \\
&\quad + (p_{k+2} - p_{k+1})^2 + p_k(p_{k+2} + p_{k+1} - 2p_k) \\
&\quad + (p_{k-1} - p_{k-2})^2 + p_k(p_{k-2} + p_{k-1} - 2p_k) \\
&= 2(p_{k+1}^2 + p_{k-1}^2 - 2p_k^2) + (p_{k+2}^2 + p_{k-2}^2 - 2p_k^2) \\
&\quad + (p_k p_{k+2} + p_{k-2} p_k - 2p_{k-1} p_{k+1}) \\
&\quad + 2(p_k p_{k+1} + p_{k-1} p_k - p_{k-2} p_{k-1} - p_{k+1} p_{k+2})\,.
\end{aligned}$$

Everywhere on the right hand sides above we see second differences, thus $\mathcal{L}$ defines a small viscosity approximation to the anharmonic chain if $\varepsilon \sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, which suggests that the triplet (3.2) ought to

emerge from the hyperbolic scaling limit. However, if $\sigma(\varepsilon) = \sigma/\varepsilon > 0$ then second derivatives appear on the right hand sides of the macroscopic equations. Of course, the equation for ρ is not modified: $\partial_t \rho = \partial_x \pi$, while $\partial_t \pi = \partial_x J(\mathfrak{I}, \rho) + \sigma \, \partial_x^2 \pi$. The energy equation turns into $\partial_t \chi = \partial_x(\pi J(\mathfrak{I}, \rho) + \sigma \partial_x(\beta^{-1} + \pi^2/2)$, where $\beta = -S'_{\mathfrak{I}}(\mathfrak{I}, \rho)$. This is almost the same as (5.1), which is not a surprise: $\mathcal{G}_{ch}$ defines an exchange mechanism, too.

We must be careful about the statements outlined above. A rigorous proof of the strong ergodic hypothesis certainly requires some additional work. The control of the stochastic integrals in the evolution equation for energy seems to be more problematic because the quadratic variation of the martingales M_k,

$$dM_k := p_k\big((p_{k+1}-p_{k-1})\,dw_k + (p_{k+2}-p_{k+1})\,dw_{k+1} + (p_{k-2}-p_{k-1})\,dw_{k-1}\big)$$

is too large: terms as $p_k^2(p_{k+1} - p_k)^2$ can not be bounded by means of the energy - entropy inequality. Anyway, we have to assume that fourth powers of the velocities have bounded expectations at time zero.

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