A Note on Quarkonial Systems and Multilevel Partition of Unity Methods

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Abstract

We discuss the connection between the theory of quarkonial decompositions for function spaces developed by Hans Triebel, and the multilevel partition of unity method. The central result is an alternative approach to the stability of quarkonial decompositions in Besov spaces $B^s_{pp}(\mathbb{R}^n)$, $s > n(1/p-1)_+$, which leads to relaxed decay assumptions on the elements of a quarkonial system as the monomial degree grows.

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1 Introduction

This note is motivated by the similarities between the concept of subatomic or quarkonial decompositions introduced by Triebel [21, 22, 23] for the study of function spaces, and numerical approximation schemes based on locally enriched partition of unity functions. Although variants of the partition of unity method (PUM) have appeared under various names (meshless particle methods, generalized finite element methods, hp-clouds, to name a few) before and independently, the PUM has formally been introduced by Babuska and Melenk [11]. Its combination with multiscale approaches is usually refered to as multilevel partition of unity method (MPUM), see Schweitzer [17]. The MPUM can be considered as a merger of spectral and multiscale approximation schemes, and offers great potential and flexibility for developing adaptive schemes necessary for large-scale modeling and computation. However, it is fair to say that the theoretical understanding of PUM and MPUM methods is not as complete as that of multiscale finite element and wavelet methods [5, 6, 12, 18], on the one hand, and of spectral methods [3], on the other. In particular, the parallel developments on quarkonial decompositions in function space theory have not been taken notice of.

We start by recalling a result by Triebel [21, 22] on the existence of quarkonial frames for Besov-Hardy-Sobolev spaces. To keep the exposition simple, and stay close to the needs of MPUM theory, we concentrate on the case of Besov spaces $B^s_p := B^s_{p,p}(\mathbb{R}^n)$ on

$\mathbb{R}^n$, $n \ge 1$, with parameters $0 < p \le \infty$, and $s > \sigma_p = \max(0, n(1/p-1))$. Various concrete but equivalent definitions of B^s_p will be given later, for this introduction it is sufficient to recall that $B^s_2 = H^s(\mathbb{R}^n)$ and $B^s_\infty = \mathcal{C}^s(\mathbb{R}^n)$ represent the classical Sobolev and Hölder-Zygmund classes, respectively. Quarkonial systems

$$\mathcal{Q} := \{q^\gamma_{j,i}(x) := w^\gamma_{j,i}(x - x_{j,i})^\gamma \phi_{j.i}(x) : \ \gamma \in \mathbb{Z}^n_+, \ i \in I_j, \ j \ge 0\} \tag{1}$$

are enumerated by three indices, the scale parameter j, the position parameter $i \in I_j$ within each scale, and the multi-index $\gamma \in \mathbb{Z}^n_+$ representing the degree of the monomial factor. They are associated with point clouds $\mathcal{X}_j = \{x_{j,i} : \ i \in I_j\}$ (approximate lattices of mesh-width $\approx 2^{-j}$), and a subordinated partition of unity (PU) $\{\phi_{j,i} : i \in I_j\}$ consisting of locally supported, sufficiently smooth functions:

$$\sum_{i \in I_j} \phi_{j,i}(x) \equiv 1, \qquad x \in \mathbb{R}^n. \tag{2}$$

"Approximate lattice of mesh-width $\approx 2^{-j}$" means that for some constants c, C we have

$$\min_{i \ne i'} \operatorname{dist}(x_{j,i}, x_{j,i'}) \ge c 2^{-j}, \qquad \cup_{i \in I_j} B_{C2^{-j}}(x_{j,i}) = \mathbb{R}^n, \tag{3}$$

and "locally supported, sufficiently smooth PU" means that $\operatorname{supp} \phi_{j,i} \subset B_{C2^{-j}}(x_{j,i})$, the $\phi_{j,i}$ are uniformly bounded on $\mathbb{R}^n$, and belong to $\mathcal{C}^{\hat{s}}$ for some positive $\hat{s} > s$, and all $i \in I_j$ and $j \ge 0$ (uniform boundedness is automatic in conjunction with (2) if we assume $\phi_{j,i} \ge 0$ which is often the case by construction). Here and in the following, c, C denote positive constants that do not depend on the involved functions and parameters j, i, γ (but may depend on other parameters such as $n, p, s, \hat{s}, \ldots$, we also allow them to change from line to line), we write $A \approx B$ if $cA \le B \le CA$, and $B_r(a) = \{x \in \mathbb{R}^n : \ |x - a| \le r\}$. Finally, the scaling factors $w^\gamma_{j,i} > 0$ are determined such that for some $\rho > 0$, we have

$$\max(2^{-js}\|q^\gamma_{j,i}\|_{B^s_p}, \|q^\gamma_{j,i}\|_{L_p}) \le C 2^{-\rho|\gamma|}, \qquad \gamma \in \mathbb{Z}^n_+ \quad (|\gamma| = \gamma_1 + \ldots + \gamma_n). \tag{4}$$

A special but prominent case of these definitions is the shift-invariant case, where $I_j = \mathbb{Z}^n$, $\mathcal{X}_j = 2^{-j}\mathbb{Z}^n$, and $\phi_{j,i}(x) = \phi(2^j x - i)$ is generated by a single, compactly supported $\phi \in B^{\hat{s}}_\infty$ satisfying

$$\sum_{i \in \mathbf{Z}^n} \phi(x - i) \equiv 1.$$

With these definitions at hand, the results from [22] (see also Theorem 1.39 in [23]) imply the following

Theorem 1 *Let $\mathcal{Q}$ be a quarkonial system as in (1) with the properties specified above (including the normalization condition (4)). Then, for any $\rho > 0$, $0 < p \le \infty$, and $\sigma_p < s < \hat{s}$, the system $\mathcal{Q}$ is stable in B^s_p, in the sense that any $f \in B^s_p$ possesses a representation*

$$f(x) = \sum_{\gamma \in \mathbf{Z}^n_+} \sum_{j=0}^{\infty} \sum_{i \in I_j} \lambda^\gamma_{j,i} q^\gamma_{j,i}(x) \tag{5}$$

(unconditional convergence in S' and $L_{\max(1,p)}$) such that

$$\|\lambda\|_{b_p^s} := \begin{cases} (\sum_{\gamma\in\mathbf{Z}_+^n}\sum_{j=0}^\infty\sum_{i\in I_j} 2^{jsp}|\lambda_{j,i}^\gamma|^p)^{1/p}, & 0<p<\infty, \\ \sup_{\gamma\in\mathbf{Z}_+^n,\, j\ge 0,\, i\in I_j} 2^{js}|\lambda_{j,i}^\gamma| & p=\infty, \end{cases} \tag{6}$$

is finite. Moreover,

$$|||f||| := \inf_{(5)} \|\lambda\|_{b_p^s} \approx \|f\|_{B_p^s} \tag{7}$$

represents an equivalent norm on B_p^s.

For simplicity, we speak of norms also for $p<1$, even though in this case we have only quasi-norms. Note that the original theorems in [21, 22, 23] cover the general case of $B_{p,q}^s$ and $F_{p,q}^s$ spaces, and use a slightly different but essentially equivalent definition of the norm (7) and the scaling coefficients $w_{j,i}^\gamma$. Actually, [22] and [23, Corollary 1.42] also state the existence of a dual system $\{\Phi_{j,i}^\gamma \in S(\mathbb{R}^n) : \gamma\in\mathbb{Z}_+^n,\ i\in I_j,\ j\ge 0\}$ such that $\lambda_{j,i}^\gamma(f) := \langle f, \Phi_{j,i}^\gamma\rangle_{S'\times S}$ yields an admissible representation (5) for f, and $\|\lambda(f)\|_{b_p^s} \approx |||f|||_p$. In other words, $\mathcal{Q}$ is a Banach frame for B_p^s in the sense of [4, 10] if $1\le p\le\infty$. However, we will not make use of the existence of a dual system, and stick to the formulated version of stability expressed by the norm equivalence (7). We should also note that similar characterizations are available for Besov-Hardy-Sobolev spaces on domains $\Omega\subset\mathbb{R}^n$ [22], and that one of the purposes of introducing quarkonial systems was their use to *define* function spaces over more general sets such as d-sets and more general quasi-metric spaces.

The relevance of the stability property for numerical purposes is as follows: If we work with numerical approximation schemes using linear combinations of the elements of a stable system, we get access to information on the coefficient sequence λ which via the norm equivalence can be related to various norms of the residual or the numerical approximants. For instance, such a posteriori estimates are handy in adaptive schemes. Moreover, in the most important case $p=2$ the frame property is equivalent to saying that a symmetric elliptic operator equation with energy space $H^s = B_2^s$ can be turned into an equivalent, infinite linear system on ℓ_2 with bounds on its generalized condition number. This property is important for the construction of fast and efficient solvers for such operator equations. Results in this direction are discussed in, e.g., [18], mostly for wavelet systems and closely related multilevel frame systems for which there is no or little redundancy in the admissible representations (5). More redundant representation systems, such as quarkonial systems, have not yet been touched to any generality.

Let us briefly introduce the fundamental ideas of PUM and MPUM. We do this for $\mathbb{R}^n$, and follow [11, 17]. The PUM is based on two ingredients. First, we assume the existence of a PU $\{\phi_i : i\in I\}$,

$$\sum_{i\in I}\phi_i(x)\equiv 1, \qquad x\in\mathbb{R}^n,$$

consisting of sufficiently smooth functions ϕ_i subordinated to a finite-overlap open cover $\{\Omega_i\}$ of $\mathbb{R}^n$, i.e., Ω_i are bounded open subsets of $\mathbb{R}^n$ such that

$$\cup_{i\in I}\Omega_i = \mathbb{R}^n, \qquad \sup_{x\in\mathbb{R}^n} \#\{i \in I:\ x \in \Omega_i\} \le C, \qquad \text{supp}\ \phi_i \subset \Omega_i.$$

Second, for each i, we choose an appropriate finite- or infinite-dimensional function space V_i on Ω_i (called local enrichment space), and define the associated PUM space as

$$V = \{\sum_{i\in I} \phi_i(x)v_i(x): \quad v_i \in V_i,\ i \in I\}.$$

In practice, the construction of the PU often starts with a point cloud $\mathcal{X} = \{x_i:\ i \in I\}$, then fixes a finite-overlap open cover $\{\Omega_i\}$ consisting of balls, rectangular domains, or direction-dependent ellipses centered at x_i, and sets

$$\phi_i(x) = (\sum_{i'\in I} \psi_{i'}(x))^{-1}\psi_i(x),$$

where the functions ψ_i are sufficiently smooth and satisfy $\text{supp}\,\psi_i = \bar{\Omega}_i$ and $\psi_i(x) > 0, \quad x \in \Omega_i$, see [7, 8, 9, 17, 24] for examples. An alternative is the use of a finite element PU consisting of Lagrange finite element nodal basis functions (e.g., piecewise linear hat functions) with respect to a simplicial partition of the domain of interest, such versions of the PUM are refered to as GFEM or XFEM (generalized or extended finite element method, see [1]). On tensor-product partitioned domains, PUs generated from B-splines are a popular choice.

Although problem-adapted local enrichment spaces V_i are definitely of interest (see [11]), in this paper we will only consider V_i generated by polynomials. The spaces V_i are typically given by an appropriate basis or generating system, the choice of which can make a big difference in practical implementations. Choosing monomials centered at x_i for generating V_i provides the link to quarkonial systems, see below. Combined with a partition-based, finite element type PU this is almost equivalent to the hp-finite element method (hp-FEM) [2], see [16] for a comprehensive treatment of the latter. Note that the approximation theory of the hp-FEM is by no means elementary.

So far, no explicit restrictions on the support sizes of the ϕ_i have been imposed, thus allowing for rather non-uniformly spaced point clouds $\mathcal{X}$ and globally non-uniform covers $\{\Omega_i\}$, this feature is also used in hp-FEM methods. From now on we restrict ourselves to quasi-uniform covers represented by approximate lattices $\mathcal{X}_j$ of mesh-width $\approx 2^{-j}$ and PUs $\{\phi_{j,i}:\ i \in I_j\}$ as specified for Theorem 1, $j \ge 0$. To indicate the dependence on j, we add the subscript j whenever necessary for clarification. Assuming quasi-uniformity does not represent a significant loss of generality, as general PUM spaces can be well-approximated by piecing together local parts of PUM spaces with quasi-uniform covers of different mesh-widths, and allows us to directly see the connection to the quarkonial case. Indeed, if we choose

$$V_{j,i}^k = \text{span}\{(x - x_{j,i})^\gamma:\ |\gamma| \le k\}, \qquad k \ge 0,$$

or the closure of $\lim_{k\to\infty} V_{j,i}^k$ in an appropriate function space as the local enrichment spaces $V_{j,i}$, then we get PUM spaces V_j^k and V_j, respectively. The sequence V_j^k is monotone with respect to k:

$$V_j^0 \subset V_j^1 \subset \ldots \subset V_j^k \subset V_j^{k+1} \subset \ldots \subset V_j. \tag{8}$$

Unfortunately, monotonicity with respect to j for fixed k,

$$V_0^k \subset V_1^k \subset \ldots \subset V_j^k \subset V_{j+1}^k \subset \ldots, \tag{9}$$

holds only in special cases, e.g., if the underlying sequence of PUs is refinable, i.e., any $\phi_{j,i}$ can be expressed as a linear combination of PU functions $\phi_{j+1,i'}$ from the next finer scale, $j \geq 0$. Note that the ladder $\{V_j^k\}_{j\geq 0}$ represents a particular instance of MPUM spaces, more general examples can be obtained by replacing k in the definition of the local enrichment spaces $V_{j,i}^k$ by a variable degree parameter $k_{j,i}$, see [17]. If (9) holds and a certain Bernstein inequality for V_j^k can be established then for $k > s$ the Besov space B_p^s coincides with an approximation space with respect to the ladder $\{V_j^k\}_{j\geq 0}$, a result that implies that the truncated and L_p normalized quarkonial system

$$\tilde{\mathcal{Q}}^k := \{\tilde{q}_{j,i}^\gamma(x) := \tilde{w}_{j,i}^\gamma (x - x_{j,i})^\gamma \phi_{j.i}(x) : \ |\gamma| \leq k, \ i \in I_j, \ j \geq 0\}, \qquad \|\tilde{q}_{j,i}^\gamma\|_{L_p} \approx 1, \tag{10}$$

is stable in B_p^s in the same sense as stated in Theorem 1, albeit with constants in the norm equivalence that depend on the value of $k > s$. This dependence of the stability bounds on k is typically algebraic, and not exponential.

This observation explains to a certain extent why the stability of the quarkonial system $\mathcal{Q}$ stated in Theorem 1 is not as surprising, as we thought initially. Indeed, since already the truncated version $\tilde{\mathcal{Q}}^{k_0}$ for $k_0 = [s] + 1$ is stable in B_p^s, there is no need in adding more elements to it at all! In other words, if we insist on including additional quarkonial functions with $|\gamma| > k_0$ into $\tilde{\mathcal{Q}}^{k_0}$, we can easily afford this by scaling them with weights sufficiently close to zero. Indeed, the influence of a term with $|\gamma| > k_0$ and small weight on the norm in (7) gets penalized since a significant contribution in (5) is then possible only with a large coefficient $\lambda_{j,i}^\gamma$ which is detrimental to realizing the infimum in (7). Due to the expected algebraic growth of the constants in the stability estimate for $\mathcal{Q}^k$, assuming exponentially decaying weights leading to (4) is more than enough, and gives room for weakening (4). As the main technical contribution of this paper, we outline an alternative proof of Theorem 1 under a weaker decay condition than (4) but at the expense of additional restrictions on the PUs. The crucial moment is to establish a reasonably sharp Bernstein estimate for the B_p^s norm of elements from V_j^k by their L_p norm. Detailed statements will be given in the next two sections, due to space limitations some results are only proved for $n = 1$. The discussion of the application of this result, its relation to the hp-FEM, and why we believe that the obtained weakening of (4) is relevant, is beyond the scope of this paper. We admit that the partial results presented here are rudimentary, and rather view them as an invitation to help shape the theory of PUM and hp-FEM further.

2 Abstract Framework

This is a more abstract version of our approach to proving Theorem 1. We restrict ourselves to the case $0 < p < \infty$, with some changes in the definitions results and proofs carry over to $p = \infty$. Let a ladder $\{V_j^k\}$ be given such that (8) is satisfied. Assume that there are $\hat{s}_p > \sigma_p$ and k_0 such that the following estimates hold:

(J) For any $f \in B_p^s$, there exists a decomposition

$$f = \sum_{j=0}^{\infty} v_j^{k_0}, \qquad v_j^{k_0} \in V_j^{k_0}, \qquad (\sum_{j=0}^{\infty} 2^{jsp} \|v_j^{k_0}\|_{L_p}^p)^{1/p} \leq C\|f\|_{B_p^s}. \tag{11}$$

Convergence of this representation is unconditional in S' and $L_{\max(p,1)}$.

(B) For all $k \geq k_0$, $j \geq 0$, $\sigma_p < t < \hat{s}_p$, and $v_j^k \in V_j^k$, we have the Bernstein-type inequality

$$\|v_j^k\|_{B_p^t} \leq CA_{k,t} 2^{jt} \|v_j^k\|_{L_p}. \tag{12}$$

Note that we require the Jackson-type inequality (11), i.e., the existence of a "good" decomposition, only for k_0, it follows from the monotonicity assumption (8) that a similar statement trivially holds for $k \geq k_0$. In (12) the constant $A_{k,t}$ incorporates the dependence on k. Instances of $\{V_j^k\}$ related to quarkonial systems for which these assumptions hold will be given below.

Proposition 1 *Let the ladder $\{V_j^k\}$ satisfy (8), (J), and (B). Then for $k \geq k_0$*

$$|||f|||_{p,s,k} := \inf_{v_j^k \in V_j^k:\, f = \sum_{j=0}^{\infty} v_j^k} (\sum_{j=0}^{\infty} 2^{jsp} \|v_j^k\|_{L_p}^p)^{1/p} \tag{13}$$

represents an equivalent norm in B_p^s, $\sigma_p < s < \hat{s}_p$. More precisely, one has

$$c|||f|||_{p,s,k} \leq \|f\|_{B_p^s} \leq C_t(1 + A_{k,t})|||f|||_{p,s,k}, \qquad f \in B_p^s, \tag{14}$$

for fixed $t \in (s, \hat{s}_p)$.

Proof. The lower bound in (14) is a direct consequence of (J) and (8). For the upper bound we use the well-known characterization of B_p^s via differences [20, Section 2.5.12]: For $0 < p < \infty$ and $\sigma_p < s < m$,

$$\|f\|_{B_p^s} := \|f\|_{L_p} + (\sum_{l=0}^{\infty} 2^{lsp} \omega_m(2^{-l}, f)_{L_p}^p)^{1/p} \tag{15}$$

can serve as the definition of the norm in B_p^s. Here, the moduli of smoothness of order m are defined by

$$\omega_m(\delta, f)_{L_p}^p = \sup_{h \in B_\delta(0)} \|\Delta_h^m f\|_{L_p}^p, \qquad \delta > 0, \qquad f \in L_p(\mathbb{R}^n).$$

In this definition, the supremum over the ball $B_\delta(0)$ can easily be replaced by appropriate integral averages, leading to equivalent moduli and other equivalent norms in B^s_p (this technical remark comes in handy when proving Jackson-type inequalities for schemes based on local polynomial approximation, see the proof of Lemma 1). Now, take any L_p convergent decomposition $f = \sum_{j=0}^\infty v^k_j$. Consider the case $1 \le p < \infty$. Then, for an arbitrarily picked $s < t < \hat{s}_p$, we have

$$\begin{aligned}
\omega_m(2^{-l}, f)_{L_p} &\le \sum_{j=0}^\infty \omega_m(2^{-l}, v^k_j)_{L_p} \\
&\le C(\sum_{j<l} 2^{-lt}\|v^k_j\|_{B^t_p} + \sum_{j\ge l} \|v^k_j\|_{L_p}) \\
&\le C(A_{k,t}\sum_{j<l} 2^{-(l-j)t}\|v^k_j\|_{L_p} + \sum_{j\ge l} \|v^k_j\|_{L_p}),
\end{aligned}$$

where in the first step the definition of the B^t_p norm for v^k_j, $j \le l$, and $\omega_m(\delta, v^k_j) \le 2^m\|v^k_j\|_{L_p}$ for $j > l$ have been used. The last step is justified by (B). For $l = 0$, the right-hand side of this inequality also majorizes the term $\|f\|_{L_p}$.

It remains to substitute into the B^s_p norm expression (15), and apply Jensen's inequality appropriately. We show the main steps of this standard procedure only for completeness:

$$\begin{aligned}
\|f\|^p_{B^s_p} &\le C\sum_{l=0}^\infty 2^{lsp}(A_{k,t}\sum_{j\le l} 2^{-(l-j)t}\|v^k_j\|_{L_p} + \sum_{j>l} \|v^k_j\|_{L_p})^p \\
&\le C_\epsilon\sum_{l=0}^\infty 2^{lsp}(A^p_{k,t}\sum_{j\le l} 2^{-(l-j)p(t-\epsilon)}\|v^k_j\|^p_{L_p} + \sum_{j>l} 2^{(j-l)p\epsilon}\|v^k_j\|^p_{L_p}) \\
&\le C_\epsilon\sum_{j=0}^\infty 2^{jsp}\|v^k_j\|^p_{L_p}(A^p_{k,t}\sum_{l\ge j} 2^{-(l-j)p(t-\epsilon-s)} + \sum_{l<j} 2^{(j-l)p(\epsilon-s)}) \\
&\le C_\epsilon(1 + A^p_{k,t})\sum_{j=0}^\infty 2^{jsp}\|v^k_j\|^p_{L_p},
\end{aligned}$$

where $0 < \epsilon < \min(s, t-s)$ can be fixed arbitrarily. Taking, e.g., $\epsilon = \min(s, t-s)/2$, the constant C_ϵ can made dependent on t (and s and p), only. Recall that the decomposition of f into a sum of $v^k_j \in V^k_j$ was arbitrary, and take the infimum over all such representations. This gives the upper estimate in (14), and concludes the argument for Proposition 1 for $1 \le p < \infty$. The case $p < 1$ is similar, the main change being the use of the inequality $\|f + g\|^p_{L_p} \le \|f\|^p_{L_p} + \|g\|^p_{L_p}$ instead of the triangle inequality for norms, Jensen's inequality is not needed.

Proposition 1 characterizes Besov spaces as approximation spaces with respect to the ladder $\{V^k_j\}_{j\ge 0}$. We can try to "refine" this result by introducing L_p stable bases or generating systems, as in the theory of finite element multilevel methods [12]. We present a version suitable for the application to quarkonial systems, and look at generating systems for the V^k_j that are hierarchical in k in the following sense. Let us denote by $\Psi_j^k = \{\psi_{j,r}^k : \ r \in I_{j,k}\}$ a system of functions belonging to V_j^k, $k \geq k_0$, such that $\|\psi_{j,r}^k\|_{L_p} \approx 1$, and the linear span of the system

$$\Phi_j^k = \cup_{l=k_0}^k \Psi_j^l \tag{16}$$

is V_j^k. We assume L_p stability properties, $0 < p < \infty$, of the system (16) as follows:

(L) For $k = k_0$, we assume that for every $v_j^{k_0} \in V_j^{k_0} \cap L_p$, there exists a decomposition

$$v_j^{k_0} = \sum_{r \in I_{j,k_0}} c_{j,r}^{k_0} \psi_{j,r}^{k_0} \quad \text{such that} \quad (\sum_{r \in I_{j,k_0}} |c_{j,r}^{k_0}|^p)^{1/p} \leq C \|v_j^{k_0}\|_{L_p}.$$

(U) For $k \geq k_0$, we have an upper bound

$$\| \sum_{l=k_0}^{k} \sum_{r \in I_{j,l}} c_{j,r}^l \psi_{j,r}^l \|_{L_p} \leq C A_k (\sum_{l=k_0}^{k} \sum_{r \in I_{j,l}} |c_{j,r}^l|^p)^{1/p}$$

for all representations with respect to Φ_j^k for which the right-hand side is finite. The possible dependence of the constant A_k on other parameters such as n, p is not shown but it must be independent of the scale parameter $j \geq 0$.

The construction in (16) is hierarchical, in the sense, that $\Phi_j^{k+1} \supset \Phi_j^k$ is obtained by adding the increment system Ψ_j^{k+1}. Based on these additional assumptions, it is easy to show, that under the conditions of Proposition 1 the spaces B_s^p can also be characterized in terms of weighted ℓ_p norms of the coefficients of decompositions with respect to the union of all Φ_j^k, $j \geq 0$, with equivalence constants depending on k via the constants $A_{k,t}$ and A_k from (B) and (U). More importantly, there is a choice of weights $\tilde{w} = \{\tilde{w}^k\}_{k \geq k_0}$ such that the weighted system

$$\Psi_{\tilde{w}} = \{\tilde{\psi}_{j,r}^k := \tilde{w}^k \psi_{j,r}^k : \quad r \in I_{j,k}, \ k \geq k_0, \ j \geq 0\} \tag{17}$$

is stable in B_p^s. Appropriate weight sequences $\tilde{w}$ can be computed explicitly, once estimates for $A_{k,t}$ and A_k are known. In the Hilbert space case $p = 2$, this has been already done in [13].

Proposition 2 *Let, in addition to the conditions of Proposition 1, assumptions (L) and (U) be satisfied.*
a) The system $\Phi^k := \cup_{j \geq 0} \Phi_j^k = \cup_{l=k_0}^k \cup_{j \geq 0} \Psi_j^l$, $k \geq k_0$, *is stable in* B_p^s *in the sense that*

$$|||f|||_{p,s,k}^* := \inf_{c_{j,r}^l : \, f = \sum_{l=k_0}^k \sum_{j=0}^\infty \sum_{r \in I_{j,l}} c_{j,r}^l \psi_{j,r}^l} (\sum_{l=k_0}^{k} \sum_{j=0}^{\infty} \sum_{r \in I_{j,l}} 2^{jsp} |c_{j,r}^l|^p)^{1/p}$$

represents an equivalent norm in B_p^s, $\sigma_p < s < \hat{s}_p$. *More precisely, one has*

$$c|||f|||_{p,s,k}^* \leq \|f\|_{B_p^s} \leq C_t (1 + A_{k,t}) A_k |||f|||_{p,s,k}^*, \qquad f \in B_p^s, \tag{18}$$

for fixed $t \in (s, \hat{s}_p)$, where the constants $A_{k,t}$ and A_k are defined by (12) and (U), respectively.
b) For an appropriate weight sequence $\tilde{w}$ depending on $A_{k,t}$ and A_k and further parameters such as p and s, the system $\Psi_{\tilde{w}}$ defined in (17) is stable in B^s_p, i.e.,

$$|||f|||^*_{p,s} := \inf_{\tilde{c}^l_{j,r}:\, f=\sum_{l=k_0}^\infty \sum_{j=0}^\infty \sum_{r\in I_{j,l}} \tilde{c}^l_{j,r}\tilde{\psi}^l_{j,r}} (\sum_{l=k_0}^\infty \sum_{j=0}^\infty \sum_{r\in I_{j,l}} 2^{jsp}|\tilde{c}^l_{j,r}|^p)^{1/p} \qquad (19)$$

represents an equivalent norm in B^s_p, $\sigma_p < s < \hat{s}_p$.

Proof. Part a) is a straightforward consequence of Proposition 1 if one substitutes (L) and (U) into the definition (13), to get lower and upper bounds of the form

$$|||f|||^*_{p,s,k} \le |||f|||^*_{p,s,k_0} \le C|||f|||_{p,s,k_0} \le C\|f\|_{B^s_p},$$

and

$$\|f\|_{B^s_p} \le C_t(1+A_{k,t})|||f|||_{p,s,k} \le C_t(1+A_{k,t})A_k|||f|||^*_{p,s,k},$$

respectively.

For proving part b), we set $\tilde{w}^{k_0} = 1$, the remaining $\tilde{w}^k > 0$ will be determined below. Then, the lower estimate

$$|||f|||^*_{p,s} \le \inf_{\tilde{c}^{k_0}_{j,r}:\, f=\sum_{j=0}^\infty \sum_{r\in I_{j,k_0}} \tilde{c}^{k_0}_{j,r}\tilde{\psi}^{k_0}_{j,r}} (\sum_{j=0}^\infty \sum_{r\in I_{j,k_0}} 2^{jsp}|\tilde{c}^{k_0}_{j,r}|^p)^{1/p} = |||f|||^*_{p,s,k_0} \le C\|f\|_{B^s_p}$$

follows from the case $k = k_0$ in part a).

The upper estimate is equally straightforward if $0 < p \le 1$. Take an arbitrary decomposition

$$f = \sum_{l=k_0}^\infty \sum_{j=0}^\infty \sum_{r\in I_{j,l}} \tilde{c}^l_{j,r}\tilde{\psi}^l_{j,r} = \sum_{l=k_0}^\infty f^l,$$

where each term

$$f^l := \sum_{j=0}^\infty \sum_{r\in I_{j,l}} (\tilde{w}^l\tilde{c}^l_{j,r})\psi^l_{j,r}$$

is represented with respect to Ψ^l and thus Φ^l, $l \ge k_0$. Again using part a), we can write

$$\begin{aligned}\|f\|^p_{B^s_p} &\le \sum_{l=k_0}^\infty \|f^l\|^p_{B^s_p} \le C\sum_{l=k_0}^\infty (1+A_{l,t})^p A_l^p \sum_{j=0}^\infty \sum_{r\in I_{j,l}} 2^{jsp}|\tilde{w}^l\tilde{c}^l_{j,r}|^p \\ &\le C(\sup_{l\ge k_0} \tilde{w}^l(1+A_{l,t})A_l)^p \sum_{l=k_0}^\infty \sum_{j=0}^\infty \sum_{r\in I_{j,l}} |\tilde{c}^l_{j,r}|^p.\end{aligned}$$

In other words, choosing an appropriate value for t, and setting

$$\tilde{w}^l := (1+A_{k_0,t})A_{k_0}((1+A_{l,t})A_l)^{-1}, \qquad l \ge k_0, \qquad (20)$$

gives the desired result.

For the case $1 < p < \infty$, one gets a similar result upon starting the estimates using

$$\|f\|_{B_p^s}^p \leq (\sum_{l\geq k_0} \|f^l\|_{B_p^s})^p \leq (\sum_{l=k_0}^{\infty} \alpha_l^{-1})^{p-1} \sum_{l\geq k_0} \alpha_l^{p-1} \|f^l\|_{B_p^s}^p,$$

where the positive sequence $\{\alpha_k\}_{k\geq k_0}$ is such that $\sum_{k=k_0}^{\infty} \alpha_k^{-1} < \infty$. Continuing as above, one gets a similar estimate, with an additional factor α_l in the definition of $\tilde{w}^l$. Since we are interested in tight estimates for the weight sequences, a slightly better result can be obtained if we split the arbitrarily given representation of f by setting

$$\tilde{c}_{j,r}^l = \sum_{k\geq l} \tilde{c}_{j,r}^{l,k}, \qquad \tilde{c}_{j,r}^{l,k} = \alpha_{l,k} \tilde{c}_{j,r}^l,$$

where the factors $\alpha_{l,k} > 0$, $\sum_{k\geq l} \alpha_{l,k} = 1$, will be chosen below in an optimal way. Then,

$$f = \sum_{k=k_0}^{\infty} \tilde{f}^k, \qquad \tilde{f}^k := \sum_{l=k_0}^{k} \sum_{j=0}^{\infty} \sum_{r\in I_{j,l}} (\tilde{w}^l \tilde{c}_{j,r}^{l,k}) \psi_{j,r}^l,$$

and, as indicated above,

$$\begin{aligned} \|f\|_{B_p^s}^p &\leq (\sum_{k=k_0}^{\infty} \alpha_k^{-1})^{p-1} \sum_{k=k_0}^{\infty} \alpha_k^{p-1} \|\tilde{f}^k\|_{B_p^s}^p \\ &\leq C \sum_{k=k_0}^{\infty} \alpha_k^{p-1} (1 + A_{k,t})^p A_k^p \sum_{l=k_0}^{k} \sum_{j=0}^{\infty} \sum_{r\in I_{j,l}} 2^{jsp} |\tilde{w}^l \tilde{c}_{j,r}^{l,k}|^p \\ &= C \sum_{l=k_0}^{\infty} \sum_{j=0}^{\infty} \sum_{r\in I_{j,l}} (\tilde{w}^l)^p \sum_{k=l}^{\infty} \alpha_k^{p-1} (1 + A_{k,t})^p A_k^p 2^{jsp} |\tilde{c}_{j,r}^{l,k}|^p \\ &= C \sum_{l=k_0}^{\infty} \sum_{j=0}^{\infty} \sum_{r\in I_{j,l}} 2^{jsp} |\tilde{c}_{j,r}^l|^p ((\tilde{w}^l)^p \sum_{k=l}^{\infty} \alpha_k^{p-1} (1 + A_{k,t})^p A_k^p \alpha_{l,k}^p). \end{aligned}$$

Next we minimize this bound by choosing, for each $j \geq 0$, $l \geq k_0$, $r \in I_{j,l}$, an appropriate sequence $\{\alpha_{l,k}\}_{k\geq l}$ satisfying the stated properties. Obviously, since for any sequence of positive numbers a_k the minimum

$$\min_{\beta_k>0:\, \sum_{k=l}^{\infty} \beta_k = 1} \sum_{k=l}^{\infty} a_k \beta_k^p = (\sum_{k=l}^{\infty} a_k^{-1/(p-1)})^{-(p-1)},$$

is achieved for a unique sequence β_k^* provided the series in the right-hand side converges (to see this, use Jensen's inequality), we can apply this with $a_k = \alpha_k^{p-1}(1 + A_{k,t})^p A_k^p$, set $\alpha_{l,k} = \beta_k^*$, and substitute the minimum value into the previous estimate. We get the desired upper stability estimate

$$\|f\|_{B_p^s}^p \leq C \sum_{l=k_0}^{\infty} \sum_{j=0}^{\infty} \sum_{r\in I_{j,l}} 2^{jsp} |\tilde{c}_{j,r}^l|^p,$$

if we set

$$\tilde{w}^l = \frac{w'_l}{w'_{k_0}}, \qquad w'_l := (\sum_{k=l}^{\infty} \alpha_k^{-1}((1+A_{k,t})A_k)^{-q})^{1/q}, \quad l \geq k_0, \quad q = p/(p-1). \tag{21}$$

Convergence of the involved series is not an issue as $A_k \geq c > 0$ and $\sum_{k=k_0}^{\infty} \alpha_k^{-1} < \infty$ is assumed. With $\{\tilde{w}^l\}_{l \geq k_0}$ defined by (20) for $0 < p \leq 1$ and by (21) for $1 < p < \infty$, we have proved part b) of Proposition 2 as well.

Remark 1. We note that under the assumption of algebraic growth in k for the constants $A_{k,t}$ and A_k, i.e., if

$$0 < c \leq (1+A_{k,t})A_k \leq Ck^{\beta}, \qquad k \geq k_0, \qquad \beta \geq 0, \tag{22}$$

then

$$w_k = \begin{cases} \frac{(1+A_{k_0,t})A_{k_0}}{(1+A_{k,t})A_k} \geq ck^{-\beta}, & 0 < p \leq 1, \\ \\ \frac{w'_k}{w'k_0} \geq c(\sum_{k \geq l} \alpha_k^{-1} k^{-q\beta})^{1/q} \geq c_\epsilon k^{-(\beta+\epsilon)}, & 1 < p < \infty, \end{cases} \tag{23}$$

for arbitrary $\epsilon > 0$ (set, e.g., $\alpha_k = k^{-1-q\epsilon}$).

3 Application to Quarkonial Systems

Now we turn to the verification of the assumptions of Proposition 1 and 2 for quarkonial systems of the form (1) associated with a point cloud $\mathcal{X}_j = \{x_{j,i} : i \in I_j\}$ and PUs $\{\phi_{j,i} \in B_\infty^{\hat{s}} : i \in I_j\}$ satisfying the properties mentioned in Section 1 (to cover certain examples as in Theorem 2 below, we replace the first condition in (3) by the weaker finite overlap condition

$$\#\{i \in I_j : \, x \in B_{C2^{-j}}(x_{j,i})\} \leq C, \qquad x \in \mathbb{R}^n. \tag{24}$$

The spaces V_j^k are generated by $\Phi_j^k := \{\tilde{q}_{j,i}^\gamma : \; i \in I_j, \; |\gamma| \leq k\}$, where $\tilde{q}_{j,i}^\gamma$ is a scaled version of $q_{j,i}^\gamma$ such that

$$\|\tilde{q}_{j,i}^\gamma\|_{L_p} \approx 1, \qquad \gamma \in \mathbb{Z}_+^n, \quad i \in I_j, \quad j \geq 0.$$

We caution the reader that this implies that the systems are scaled differently for different values of $p \in (0,\infty)$. Note that (8) is automatically satisfied, and that by identifying, for given k_0, the systems $\Psi_j^{k_0}$ with $\Phi_j^{k_0}$, and Ψ_j^k with $\{\tilde{q}_{j,i}^\gamma : \; i \in I_j, \; |\gamma| = k\}$ for $k > k_0$, respectively, we are in the setting covered by Proposition 1 and 2. In the process of verifying the conditions (J), (B), (L), and (U), we will introduce further restrictions on the PUs. These restrictions will be briefly motivated, and further commented on in the next section. Let us first deal with (J) and (L) which concern the derivation of lower bounds, and involve properties of the ladder $\{V_j^{k_0}\}_{j \geq 0}$ for some fixed k_0.

Lemma 1 *Let $0 < p < \infty$, $\sigma_p < s < \hat{s}$, and assume that the ladder $\{V_j^k\}_{j \geq 0}$ satisfies (9). Then (J) holds for any $k_0 > s - 1$.*

Proof. We sketch the standard argument. Let $Q_{j,i}$ be the cube of side-length $C2^{-j+1}$ centered at $x_{j,i}$ containing the ball $B_{C2^{-j}}(x_{j,i}) \supset \operatorname{supp} \phi_{j,i}$, see (2). According to (2), the family $Q_{j,i}$, $i \in I_j$, has finite overlap, i.e., $\operatorname{card}\{i \in I_j : x \in Q_{j,i}\} \le C$ for some constant independent on j. According to Whitney's theorem, for any $f \in L_p$, there exists a polynomial of degree k_0 such that

$$\|f - P_{j.i}\|_{L_p(Q_{j,i})} \le C\omega_{k_0+1}(2^{-j}, f)_{L_p(Q_{j,i})}, \qquad i \in I_j,\ j \ge 0.$$

Thus, the functions $u_j^{k_0} := \sum_{i\in I_j} \phi_{j,i} P_{j,i} \in V_j^{k_0}$, satisfy

$$\begin{aligned} \|f - u_j^{k_0}\|_{L_p}^p &= \int_{\mathbf{R}^n} |\sum_{i\in I_j} \phi_{j,i}(x)(f(x) - P_{j,i}(x))|^p \, dx \\ &\le C \sum_{i\in I_j} \|f - P_{j,i}\|_{L_p(Q_{j,i})}^p \le C \sum_{i\in I_j} \omega_{k_0+1}(2^{-j}, f)_{L_p(Q_{j,i})}^p \\ &\le C\omega_{k_0+1}(2^{-j}, f)_{L_p}^p. \end{aligned}$$

Here we have used the finite overlap property of the $Q_{j,i}$ several times, the uniform boundedness of the PU functions $\phi_{j,i}$, and properties of the moduli of smoothness on cubes, in order to formally facilitate the last step of the estimation. More precisely, the inequality

$$\omega_m(\delta, f)_{L_p(Q)}^p \approx \delta^{-m} \int_{B_\delta(0)} \|\Delta_h^m\|_{L_p(Q_{m,h})}^p \, dh, \qquad 0 < \delta < 1/(2m),$$

valid for the unit cube $Q = [0,1]^n$, is sufficient for this purpose. This type of reasoning is at the core of proving approximation results for the PUM, see [11, 17], it does not require the monotonicity assumption (9).

The latter is needed only to show that the terms $v_j^{k_0}$ in the telescopic sum representation

$$f = \sum_{j=0}^{\infty} v_j^{k_0}, \qquad v_0^{k_0} = u_0^{k_0},\ v_j^{k_0} = u_j^{k_0} - u_{j-1}^{k_0}, \quad j \ge 1,$$

also belong to $V_j^{k_0}$. The rest is standard:

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{jsp} \|v_j^{k_0}\|_{L_p}^p &\le C(\|f\|_{L_p}^p + \sum_{j=0}^{\infty} 2^{jsp} \|f - u_j^{k_0}\|_{L_p}^p) \\ &\le C(\|f\|_{L_p}^p + \sum_{j=0}^{\infty} 2^{jsp} \omega_{k_0+1}(2^{-j}, f)_{L_p}^p) \le C\|f\|_{B_p^s}^p, \end{aligned}$$

where the last step requires $k_0 > s - 1$ and follows from the definition (15). This proves Lemma 1. Whether the monotonicity assumption (9) can be removed or significantly weakened is open.

Next we consider property (L). Without further conditions, it may easily fail for V_j^k generated by a generically constructed quarkonial system. The failure of L_p stability for generating systems in PUM spaces is one of the drawbacks of the method, see [1] for the discussion of alternatives. For example, look at a shift-invariant construction for $\mathbb{R}^1$, where $\phi(x) = (1 - |x|)_+$ is the hat function, and set $k_0 = 1$. The resulting V_j^1 is a subspace of the space of quadratic C^0 splines with knot sequence $\mathcal{X}_j = 2^{-j}\mathbb{Z}$, and consists of all functions

$$\Phi_j^1 = \{\tilde{q}_{j,i}^0(x) = 2^{j/p}\phi(2^j x - i),\;\; \tilde{q}_{j,i}^1(x) = 2^{j/p}(2^j x - i)\phi(2^j x - i)\}_{i\in\mathbf{Z}}$$

is a basis in V_j^1 which is not L_p stable: Consider

$$v_{j,N}^1 = \sum_{i=1}^{N} \tilde{q}_{j,i}^1 \in V_j^1, \qquad N \to \infty,$$

and check that $\|v_{j,N}^1\|_{L_p} \approx 1$ while the ℓ_p norm of the coefficients grows as $N^{1/p}$.

The problem with this example is that the basis Φ_j^1 fails to be locally linear independent in V_j^1. This is a useful notion for PUM spaces, see [7] for a discussion. We give here only simple sufficient conditions for (L) to hold.

Lemma 2 *a) Let Φ_j^0 be an L_p stable, normalized basis in its span V_j^0, i.e.,*

$$c(\sum_{i\in I_j}|c_i|^p)^{1/p} \le \|\sum_{i\in I_j} c_i\tilde{\phi}_{j,i}\|_{L_p} \le C(\sum_{i\in I_j}|c_i|^p)^{1/p},$$

with constants independent of j, where $\tilde{\phi}_{j,i} = \tilde{q}_{j,i}^0$ are the L_p normalized PU functions. Then (L) holds for $k_0 = 0$.
b) Let the PUs $\{\phi_{j,i} :\; i \in I_j\}$ have small overlap, i.e., for each $i \in I_j$ and $j \ge 0$, we have $\phi_{j,i}(x) = 1$ on a certain ball $B_{c2^{-j}}(y_{j,i})$. Then (L) holds for any $k_0 \ge 0$, albeit with constants exponentially depending on k_0, if we take $\Phi_j^{k_0}$ as generating set $\Psi_j^{k_0}$ in $V_j^{k_0}$.

Proof. Part a) is obvious, we have included it because it is often satisfied for PUs based on spline and finite element constructions. E.g., it holds for the PUs based on linear finite element functions such as in the above counterexample. For part b), the statement follows since on each ball $B_{c2^{-j}}(y_{j,i}) \subset \mathrm{supp}\phi_{j,i} \subset B_{C2^{-j}}(x_{j,i})$ any $v_j^{k_0}$ coincides with the polynomial $p_i = \sum_{|\gamma|\le k_0} c_{j,i}^\gamma \tilde{q}_{j,i}^\gamma$ of degree $\le k_0$. This allows us to recover the coefficients $c_{j,i}^\gamma$, $|\gamma| \le k_0$, and prove that

$$\sum_{|\gamma|\le k_0} |c_{j,i}^\gamma|^p \le C\tilde{A}_{k_0}^p \|p_i\|_{B_{c2^{-j}}(y_{j,i})}^p = C\tilde{A}_{k_0}^p \|v_j^{k_0}\|_{B_{c2^{-j}}(y_{j,i})}^p.$$

That the constants in this estimate can be chosen independently of j and i (but not of k_0) follows from $1 \approx \|\tilde{q}_{j,i}^\gamma\|_{L_p} \approx \|\tilde{q}_{j,i}^\gamma\|_{B_{c2^{-j}}(y_{j,i})}$ and a compactness argument. Summation with respect to $i \in I_j$ gives the statement.

In contrast to (L), the verification of (U) is easy, at least if we are not after best possible constants, and is solely based on the finite overlap property of Φ_j^k. Indeed, simple counting based on the assumption (24) yields

$$\#\{(i,\gamma) :\; x \in \mathrm{supp}\,\tilde{q}_{j,i}^\gamma,\; i \in I_j,\; |\gamma| \le k\} \le Ck^n.$$

Thus, because of the assumed normalization $\|\tilde{q}^\gamma_{j,i}\|_{L_p} \approx 1$ we have

$$\begin{aligned}\| \sum_{|\gamma|\le k} \sum_{i\in I_j} c^\gamma_{j,i}\tilde{q}^\gamma_{j,i}\|^p_{L_p} &\le Ck^{n(\max(p,1)-1)} \sum_{|\gamma|\le k} \sum_{i\in I_j} \|c^\gamma_{j,i}\tilde{q}^\gamma_{j,i}\|^p_{L_p} \\ &\le Ck^{n(\max(p,1)-1)} \sum_{|\gamma|\le k} \sum_{i\in I_j} |c^\gamma_{j,i}|^p.\end{aligned}$$

Since $n(\max p,1)-1)/p = n(1-1/p)_+$, we have proved

Lemma 3 *The quarkonial system Φ^k_j satisfies (U) with constant $A_k = k^{n(1-1/p)_+}$.*

Finally, we turn to the Bernstein inequality (B). Our conjecture is that, at least for $1 \le p \le \infty$ and under generic assumptions on the PUs, it holds with a constant $A_{k,s} = (k+1)^{2s}$ but currently we are able to establish such results only for PUs given by piecewise polynomial functions. To keep the technical details to a minimum, consider $n = 1$, and assume that the PU functions $\phi_{j,i}$, $i \in I_j$, are C^R B-splines of some fixed degree $K \ge R+1$ with respect to a quasi-uniform partition $\mathcal{T}_j$ of mesh-width $\approx 2^{-j}$ (as $x_{j,i}$, take any point in the support of $\phi_{j,i}$). Then V^k_j, $k \ge 0$, consists of C^R splines of degree $K+k$ with respect to the same $\mathcal{T}_j$. Here $R \ge -1$ is integer, meaning that for $R = -1$ there are no global smoothness requirements, and we talk about non-smooth, piecewise polynomials of degree $K+k$.

Lemma 4 *Let V^k_j, $j \ge 0$, be the C^R spline spaces of degree $\tilde{k} := K+k$ over quasi-uniform partitions $\mathcal{T}_j$ of $\mathbb{R}$ with mesh-width $\approx 2^{-j}$, as described before. Then, for all $v^k_j \in V^k_j \cap L_p$, we have*

$$\omega_{R+2}(\delta, v^k_j)_{L_p} \le \begin{cases} \min(2^{R+2}, C(\tilde{k}^2 2^j\delta)^{R+1+1/p}), & 1 \le p < \infty, \\ \min(2^{(R+2)/p}, C\tilde{k}^{(R+2)/p}(2^j\delta)^{R+1+1/p}), & 0 < p < 1. \end{cases} \qquad (25)$$

and the following Bernstein-type inequality holds for $0 < s < R+1+\min(1,1/p)$:

$$\|v^k_j\|_{B^s_p} \le C\tilde{k}^{2s(R+2)/(R+1+\max(1,1/p))} 2^{js} \|v^k_j\|_{L_p}. \qquad (26)$$

Proof. Let $1 \le p < \infty$, set without loss of generality $j = 0$, and consider an arbitrary $v \in V^k_0 \cap L_p(\mathbb{R})$. If $h \ge c\tilde{k}^{-2}$ for some fixed c, then the inequality (25) follows from

$$\|\Delta^{R+2}_h v\|_{L_p} \le 2\|\Delta^{R+1}_h v\|_{L_p} \le \ldots \le 2^{R+2}\|v\|_{L_p}.$$

For $0 < h < c(\tilde{k}-(R+1))^{-2}$, we use $v \in W^{R+1}_p(\mathbb{R})$ and

$$\|\Delta^m_h f\|_{L_p} = \left(\int_{\mathbb{R}} \left| \int_x^{x+h} \Delta^{m-1}_h f'(s)\,ds \right|^p dx \right)^{1/p} \le h\|\Delta^{m-1}_h f'\|_{L_p}, \quad f \in W^1_p(\mathbb{R}), \qquad (27)$$

recursively $(R+1)$-times:

$$\|\Delta^{R+2}_h v\|_{L_p} \le h^{R+1}\|\Delta_h v^{(R+1)}\|_{L_p},$$

where $v^{(R+1)}$ is now a piecewise polynomial function of degree $\tilde{k}-(R+1)$ with respect to $\mathcal{T}_0$. If we denote the intervals in this partition by $J_i=[x_i,x_{i+1}]$ and by $P_i=v|_{J_i}$ the associated polynomials of degree $\tilde{k}$ then

$$\begin{aligned}\|\Delta_h v^{(R+1)}\|_{L_p}^p &\le \sum_i \Big(\int_{x_i-h}^{x_i} |\Delta_h v^{(R+1)}(x)|^p\,dx + \int_{x_i}^{x_{i+1}} |\Delta_h v^{(R+1)}(x)|^p\,dx\Big)\\ &\le C\sum_i \big(\|P_i^{(R+1)}\|_{L_p([x_i,x_i+h])}^p + \|P_i^{(R+1)}\|_{L_p([x_{i+1}-h,x_{i+1}])}^p + h^p\|P_i^{(R+2)}\|_{L_p(J_i)}^p\big).\end{aligned}$$

Since the intervals J_i are of length ≈ 1, standard inequalities for polynomials (see paragraphs 4.8.72 and 4.9.6 in [19]) imply for the given range of h that

$$\|P_i^{(R+1)}\|_{L_p([x_i,x_i+h])}^p + \|P_i^{(R+1)}\|_{L_p([x_{i+1}-h,x_{i+1}])}^p \le Ch(\tilde{k}-(R+1))^2\|P_i^{(R+1)}\|_{L_p(J_i)}^p. \quad (28)$$

Now apply the L_p Markov inequality [15] for polynomials P of degree m,

$$\|P'\|_{L_p(J_i)} \le C(x_{i+1}-x_i)^{-1}m^2\|P\|_{L_p(J_i)}, \quad (29)$$

to the terms $\|P_i^{(R+1)}\|_{L_p(J_i)}$ and $\|P_i^{(R+2)}\|_{L_p(J_i)}$, and substitute in the previous estimate:

$$\begin{aligned}\|\Delta_h^{R+2}v\|_{L_p}^p &\le Ch^{(R+1)p}\Big(h(\tilde{k}-(R+1))^2\prod_{l=0}^{R}(\tilde{k}-l)^{2p} + h^p\prod_{l=0}^{R+1}(\tilde{k}-l)^{2p}\Big)\sum_i \|P_i\|_{L_p}^p\\ &\le C(h\tilde{k}^2)^{(R+1)p+1}\|v\|_{L_p}^p, \qquad 0<h<c\tilde{k}^{-2}.\end{aligned}$$

If one takes the supremum for $0<h\le\delta$, this covers the case $1\le p<\infty$ and $j=0$ of (25), the case of arbitrary j follows by dilation. What concerns $0<p<1$, the proof is similar. The triangle inequality is to be substituted by $\|f+g\|_{L_p}^p \le \|f\|_{L_p}^p + \|g\|_{L_p}^p$, resulting in the term $2^{(R+2)/p}$ instead of 2^{R+2}. The estimation (27) cannot hold for general f, one has to use the fact that v (and its derivatives) are piecewise polynomials for which a Nikolski-type inequality [19, Section 4.9.6] gives, e.g.,

$$\begin{aligned}&\int_{\mathbf{R}}\left|\int_x^{x+h} v(s)\,ds\right|^p dx \le \int_{\mathbf{R}} (\|v\|_{L_1(x,x+h)})^p\,dx\\ &\le (Ch^{1-1/p}\tilde{k}^{2(1/p-1)})^p\int_{\mathbf{R}}\|v\|_{L_p(x,x+h)}\,dx = Ch^p\tilde{k}^{2(1-p)}\|v\|_{L_p}^p.\end{aligned}$$

With these changes, the statement of (25) in Lemma 4 can be established also for $0<p<1$.

Finally, (26) follows from (25) as follows. Let first $1\le p<\infty$, choose $j_0\ge j$ such that $2^{j_0}\approx 2^j\tilde{k}^2$, and substitute (25) into the definition (15) of the Besov spaces for $m=R+2\ge R+1+1/p>s>0$:

$$\begin{aligned}\|v_j^k\|_{B_p^s}^p &\le C\|v_j^k\|_{L_p}^p\Big(1+\sum_{l\le j_0}2^{lsp} + \tilde{k}^{2((R+1)p+1)}\sum_{l>j_0}2^{lsp}2^{(j-l)((R+1)p+1)}\Big)\\ &\le C\|v_j^k\|_{L_p}^p\big(1+2^{j_0sp}+2^{j_0sp}(\tilde{k}2^{j-j_0})^{(R+1)p+1}\big)\\ &\le C2^{j_0sp}\|v_j^k\|_{L_p}^p \le C2^{jsp}\tilde{k}^{2sp}\|v_j^k\|_{L_p}^p,\end{aligned}$$

where the definition of j_0 and $0 < s < R+1+1/p$ have been used. The case $0 < p < 1$ is similar, now $j_0 \geq j$ needs to be taken such that $2^{j_0} \approx 2^j \tilde{k}^{2(R+2)/(R+1+1/p)}$. These details are left to the reader. Lemma 4 is proved.

The following result is an immediate corollary of Lemmas 1-4 and Proposition 2. It shows an instance, where the necessary scaling grows algebraically in the polynomial degree, and not exponentially as in (4).

Theorem 2 *Under the conditions of Lemma 4, i.e., for $n = 1$ and PUs $\{\phi_{j,i} : i \in I_j\}$ consisting of C^R B-splines of degree K w.r.t. to a sequence of nested quasi-uniform partitions $\mathcal{T}_j$ of mesh-width $\approx 2^{-j}$, the scaled quarkonial system*

$$\tilde{\mathcal{Q}} := \{k^{-\alpha}\tilde{q}^k_{j,i} : \; i \in I_j, \; j,k \geq 0\},$$

where $\tilde{q}^k_{j,i}(x) = \tilde{w}^k_{j,i}(x)(x - x_{j,i})^k\phi_{j,i}$ is L_p normalized ($\|\tilde{q}^\gamma_{j,i}\|_{L_p} \approx 1$), is stable in B^s_p for $0 < s < R + 1 + \min(1, 1/p)$ if

$$\alpha > \begin{cases} 2s + 1 - 1/p, & 1 \leq p < \infty, \\ 2s(R+2)/(R+1+1/p), & 0 < p < 1. \end{cases} \tag{30}$$

In other words, for the indicated range of s and α

$$\|f\|_{B^s_p} \approx |||f|||^{**}_{p,s,\alpha} := \inf_{c^k_{j,i} :\, f = \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i \in I_j} c^k_{j,i}\tilde{q}^k_{j,i}} \left(\sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i \in I_j} k^{\alpha p} 2^{jsp} |c^k_{j,i}|^p\right)^{1/p}. \tag{31}$$

Proof. Since the introduced B-spline PUs are refinable, (9) holds and Lemma 1 applies for any given $k_0 > s - 1$ (in this case, due to available results for spline approximation, the Jackson-type estimate (J) also holds with $k_0 = 0$ if $0 < s < K + 1$ which covers the range of s considered in Theorem 2). Lemma 2 also holds with $k_0 = 0$ since B-spline PUs are L_p-stable. Lemma 3 holds with constant $A_k = k^{(1-1/p)_+}$. Finally, we have the Bernstein estimate (26) with a constant

$$A_{k,t} = k^{2t(R+2)/(R+1+\max(1,1/p))}, \qquad 0 < t < R + 1 + \min(1, 1/p).$$

Altogether Theorem 2 follows from Proposition 2 and Remark 1, the norm equivalence (31) is a reformulation of (19), the weights $k^{-\alpha}$ in the definition of $\tilde{\mathcal{Q}}$ have been included in the coefficient norm.

Remark 2. We do not claim that the parameter range (30) is optimal, nor that the conditions of Lemmas 1-4 are final. In particular, it would be interesting to extend Lemma 4 to PUs not consisting of piecewise polynomial functions. Even in the shift-invariant case, where the PUs are generated by the shifts and dilates of a single compactly supported (refinable and sufficiently smooth) function ϕ, a statement is missing. Another possible improvement concerns Lemma 3, and consists in a more careful choice of the generating systems in the polynomial spaces: Instead of the ill-conditioned system of monomials $\{(x - x_{j,i})^\gamma, \; \gamma \in \mathbb{Z}^n_+\}$, other sets should be considered, see [16] for the hp-FEM case.

Remark 3. For future applications to design adaptive MPUM (see [17]) based on stable quarkonial systems, it is desirable to work out the case $p = 2$ in greater detail. A first step has been done in [14] in a special case (PUs consisting of characteristic functions, and quarkonial functions generated by Legendre polynomials). An additional challenge is to modify the construction of quarkonial systems such that moment conditions of sufficiently high order hold for the $\tilde{q}^{\gamma}_{j,i}$ at least for $j \geq 1$. One option is to proceed in analogy to the modifications proposed by Triebel [21, 22] for covering the range $s \leq \sigma_p$, another one to use lifting techniques as explored in wavelet theory.

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